방사광 가속기
빔 물 리

고려대학교 대학원
가속기과학과

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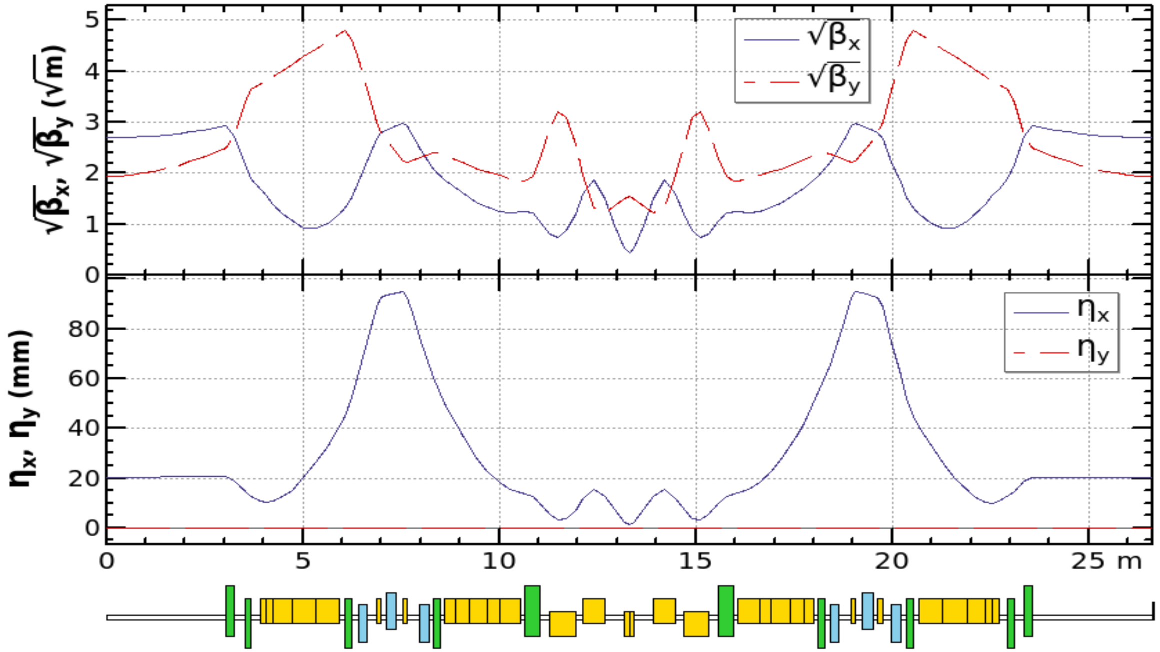
2022년 2학기 가속기 학점교류 과목
(고려대 가속기과학과 개설)

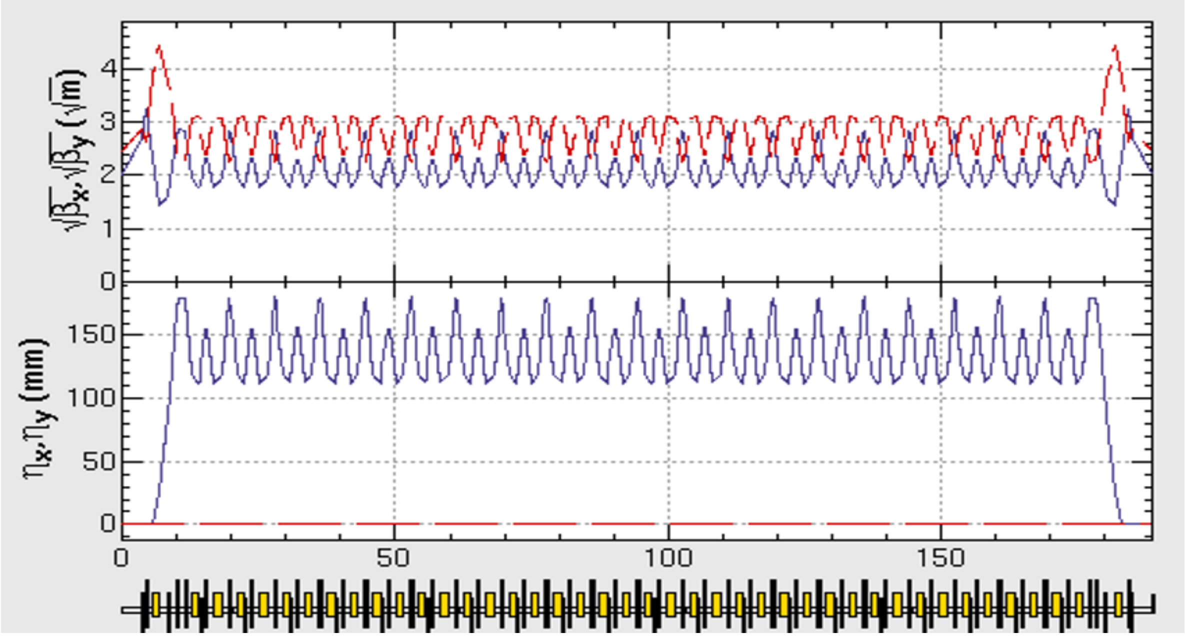
- 선형 가속기
- 빔 역학
- 가속기 실험
- 가속기 물리학
- 가속기 물리학 특론
- 빔진단 및 제어
- 전산 가속기 물리
- RF 시스템

빔 물리 연구 분야

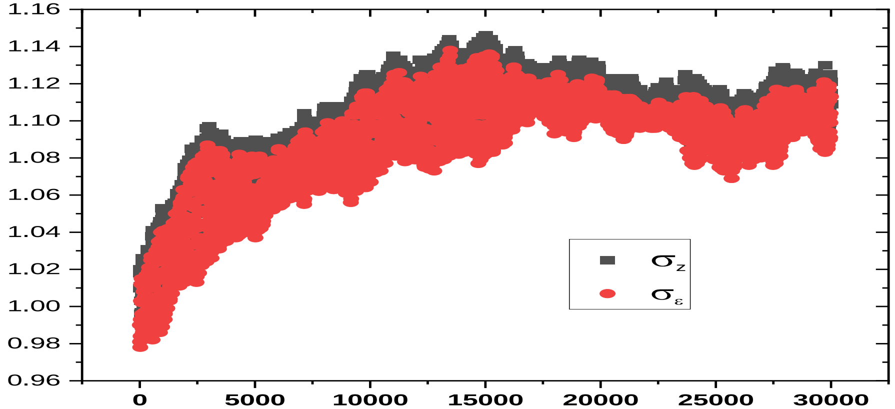
저장링 HMBA lattice 설계



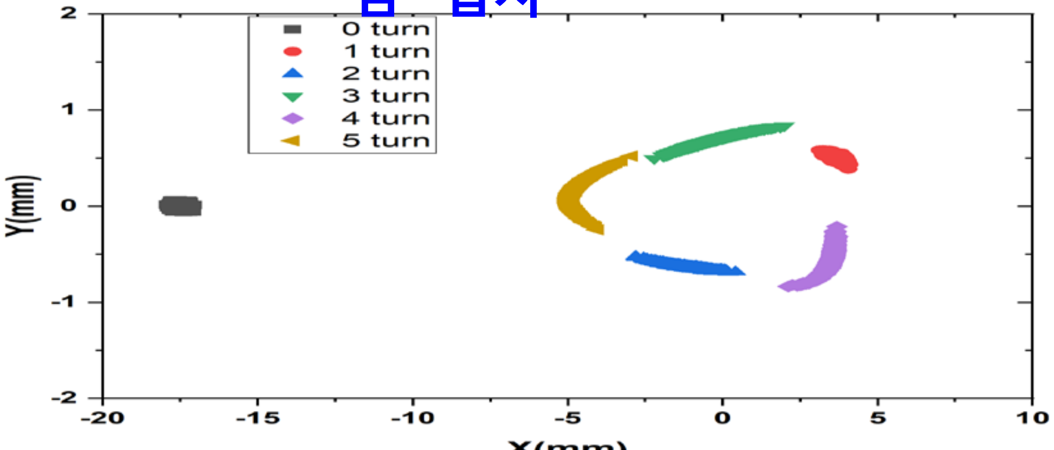
부스터링 lattice 설계



빔 불안정성



빔 입사



Contents

- Betatron motion
  : equation of motion, transfer matrix, FODO cell, chromaticity, emittance, adiabatic damping, transition energy, dispersion function, field errors

- Longitudinal motion
  : equation of motion, energy aperture, separatrix

- Radiation damping in betatron and longitudinal motion
  : energy loss, damping times

- Effects of Undulator on beam parameters
  : particle trajectory, energy spread, momentum compaction factor

Equation of motion in radial coordinates

Let's consider one particle's trajectory:

$$F = m\frac{d^2\rho}{dt^2} = -m\frac{v^2}{\rho} + evB_y \qquad (\rho = \text{상수})$$

$$F_{centrifugal} = -m\rho\omega^2 = -m\,\frac{v^2}{\rho}$$

$$F_{Lorentz} = eB_y v \qquad \rightarrow \quad \boxed{\frac{p}{e} = B_y\rho}$$

Beam rigidity [T•m]

For a general trajectory:

$$\rho \rightarrow \rho + x: \quad F = ma_r \quad \rightarrow \quad m\left[\frac{d^2}{dt^2}(\rho + x) - \frac{v^2}{\rho + x}\right] = -eB_y v \quad (\text{전자 :-e}$$

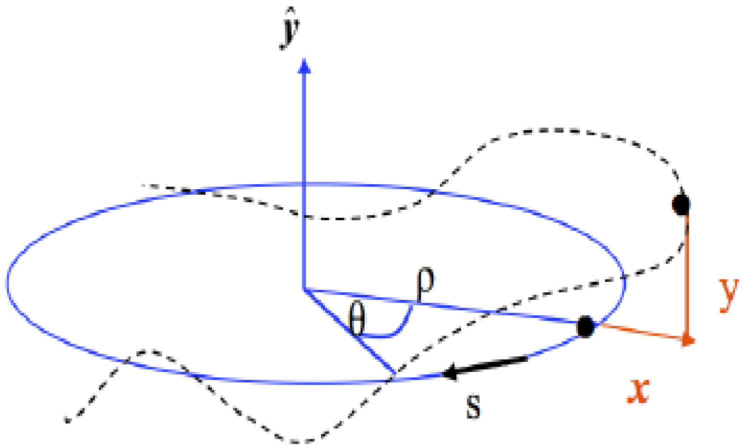
$\rho \gg x$

$$m\frac{d^2}{dt^2}(\rho + x) = m\frac{d^2}{dt^2}x \qquad\qquad m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -eB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x} + y\frac{\partial B_y}{\partial y}$ ∇•B=0

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\}$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{evB_0}{m} - x\frac{evg}{m} \qquad \left(g = \frac{\partial B_y}{\partial x}\right)$$



s=ct

Equation of motion in radial coordinates

Independent variable: t $\rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x'v$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(x'v) = \frac{d}{ds}\frac{ds}{dt}(x'v) = \frac{d}{ds}(x'v^2) = x''v^2 + x'2v\frac{dv}{ds}$$

0, 수직

$$x''v^2 - \frac{v^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{evB_0}{m} - x\frac{evg}{m}$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_0}{mv} - x\frac{eg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{\frac{p}{e}} - \frac{xg}{\frac{p}{e}} \qquad \left(\frac{g}{\frac{p}{e}} = k\right)$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{1}{\rho} - kx$$

$$\boxed{x'' + x\left(\frac{1}{\rho^2} + k\right) = 0}$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \leftrightarrow -k$ quadrupole field changes sign

$$\boxed{y'' - ky = 0}$$

Solution of the trajectory equations: focusing qudrupole

Definition:

Horizontal plane $K = \frac{1}{\rho^2} + k$

vertical plane $K = -k$

$$x'' + Kx = 0$$

General solution, for $K > 0$:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

Boundary conditions:

$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

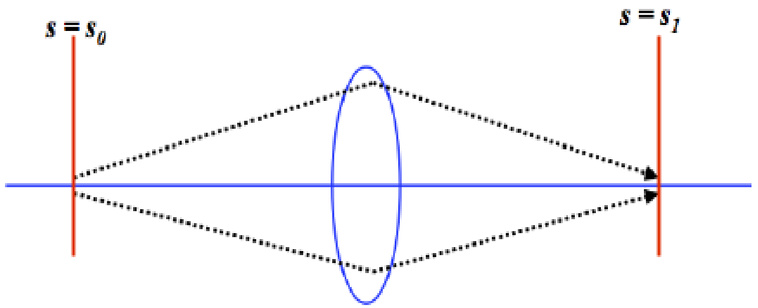
Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos(\sqrt{K}s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

$$x'(s) = -x_0\sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s)$$

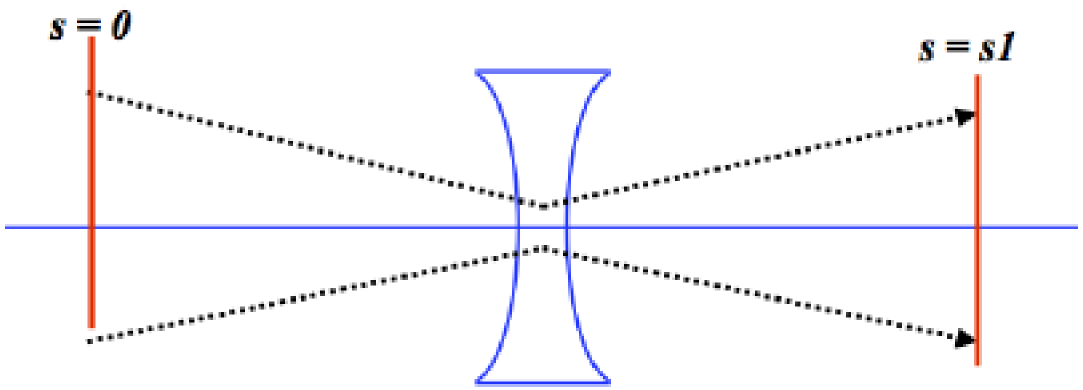
Matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}s) \\ -\sqrt{K} \sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

Solution of the trajectory equations: defocusing qudrupole



Equation of motion is

$$x'' + Kx = 0 \quad \text{with } K < 0$$

Solution is in the form:

$$x(s) = a_1 \cosh\left(\sqrt{|K|}s\right) + a_2 \sinh\left(\sqrt{|K|}s\right)$$

Transfer matrix:

$$M_{defoc} = \begin{pmatrix} \cosh\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{K}} \sinh\left(\sqrt{|K|}s\right) \\ -\sqrt{K} \sinh\left(\sqrt{|K|}s\right) & \cosh\left(\sqrt{|K|}s\right) \end{pmatrix}$$

For a drift space, i.e. when $K = 0$ $\rightarrow$ $M_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Thin-lens approximation

When the focal length, f, of the lens is much bigger than the length of the magnet L

$$f = \frac{1}{K \cdot L} \gg L$$

We can derive the limit for $L \to 0$ while keeping constant f, i.e. $K \cdot L = const$.

Transfer matrices are

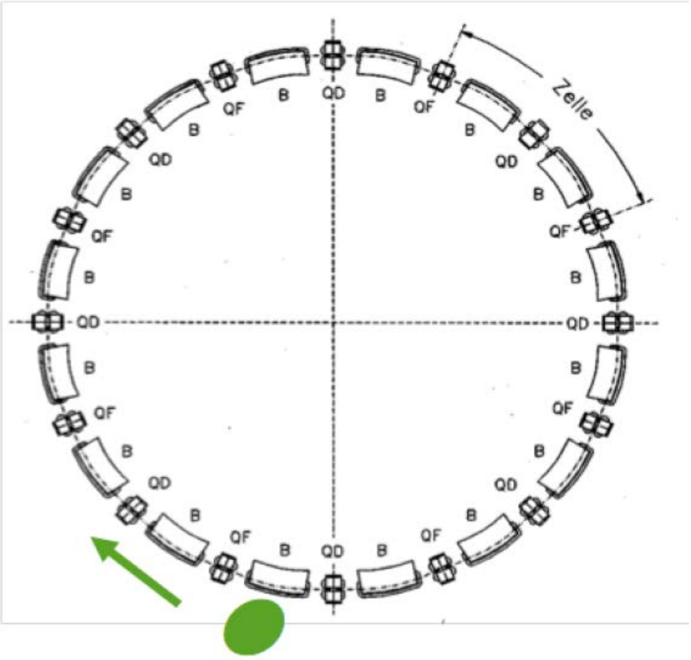
$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Focusing and defocusing magnets respectively.

Transformation through a system of lattice elements

One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M_{s1 \rightarrow s2} \cdot M_{s0 \rightarrow s1} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s0}$$



Hill's equation

Differential equation for "motions with periodic focusing properties"

$$x''(s) + K(s)x(s) = 0$$

With
- A restoring force $\neq$ const
- K(s) depends on the position s
- K(s+L) = K(s) periodic function, where L is the "lattice period"

We expect a solution in the form of a quasi harmonic oscillation : amplitude and phase will depend on the position s in the ring.

Beta function

General solution of Hill's equation: $x''(s) + K(s)x(s) = 0$

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s) + \phi) \qquad (1)$$

ε, ϕ = integration constants determined by initial conditions

$\beta(s)$ is a periodic function given by the focusing properties of the lattice ↔ quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting Eq. (1) in the equation of motion,

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$\psi(s)$ is the "phase advance" of the oscillation between the points 0 and s.

For one complete revolution, $\psi(s)$ is the number of oscillations per turn, the "tune"

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

Courant-Snyder invariant

General solution of the Hill's equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\phi) & (1) \\ x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\varphi(s)+\phi)+\sin(\varphi(s)+\phi)\} & (2) \end{cases}$$

From Eq. (1)

$$\cos(\varphi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\sqrt{\beta(s)}}$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\beta'(s)^2}{\beta(s)}$$

Insert into Eq. (2) and solve for ε

$$\varepsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

- ε is a constant of the motion, in dependent of s : Courant-Snyder invariant
- it is a parametric representation of an ellipse in the $x\ x'$ space
- the shape and the orientation of the ellipse are given by $\alpha, \beta, \gamma\ \rightarrow$ Twiss parameters

Courant-Snyder invariant

Liouville : in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

The area of ellipse, $\pi \cdot \varepsilon$, is an intrinsic beam parameter and cannot be changed by the focal properties.

Phase-space ellipse

Given the particle trajectory:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s) + \phi)$$

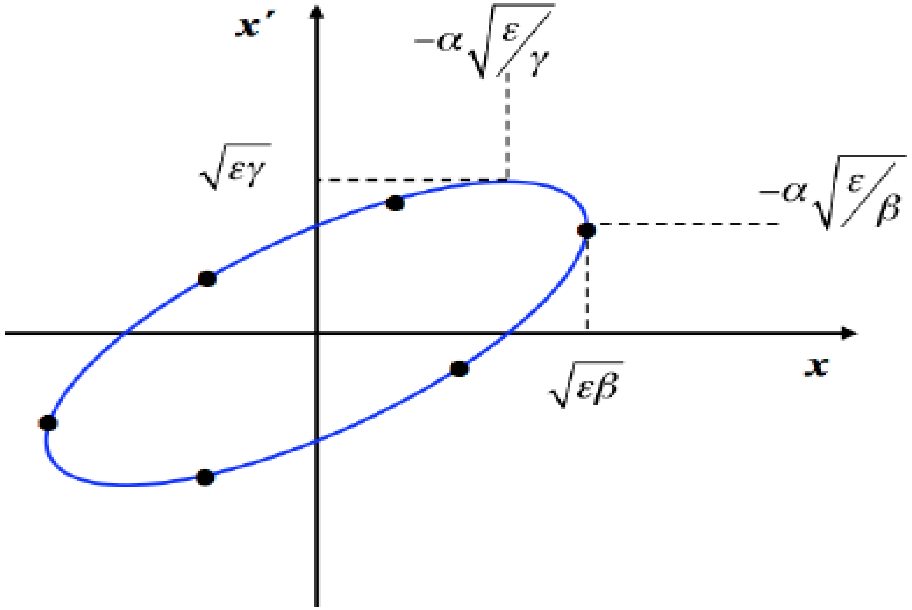
- the maximum amplitude is:

$$\hat{x}(s) = \sqrt{\varepsilon\beta}$$

- the corresponding angle, in $\hat{x}(s)$, and solving for x':

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta(s)x'^2$$

$$\rightarrow \hat{x}' = -\alpha\sqrt{\frac{\varepsilon}{\beta}}$$

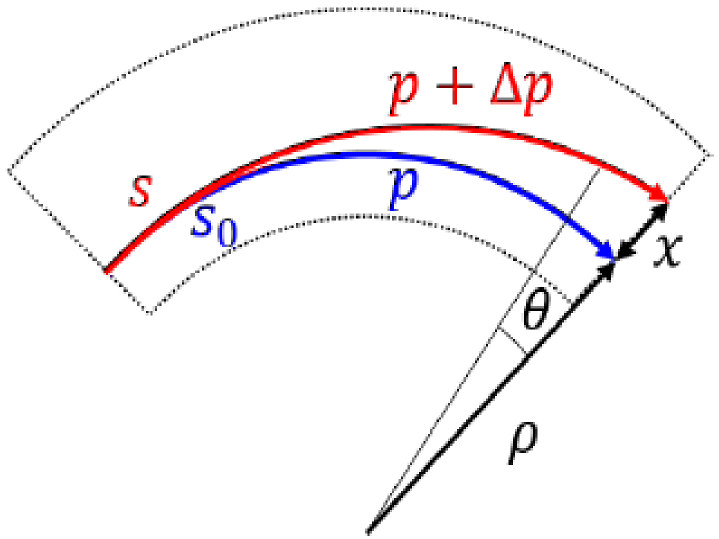


- A large β corresponds to a large beam size and a small beam divergence
- wherever β reaches a maximum or a minimum, α = 0 (and x' = 0)

Momentum Compaction factor

Off-Energy Particles

If a particle is slightly off the design momentum it will have a different orbit.



- Path length of an orbit displaced by x

$$ds_0 = \rho d\theta \qquad ds = (\rho + x)d\theta$$

- Relative difference in path length (η_x = dispersion)

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{\eta_x}{\rho}\frac{dp}{p}$$

- Integrating leads to the total path length change

$$\Delta C = \oint dl = \oint \frac{x}{\rho(s_0)}\, ds_0 = \oint \frac{\eta_x(s_0)}{\rho(s_0)}\frac{dp}{p}\, ds_0$$

since η_x is usually positive, total path length increases for higher energy particles.

Momentum compaction factor

$$\alpha_c \equiv \frac{\frac{dL}{L}}{dp} = \frac{1}{L}\oint \frac{\eta_x(s_0)}{\rho(s_0)}\frac{dp}{p}\, ds_0$$

Transition Energy

- Off-momentum particles have different revolution frequencies to on-momentum particles due to different orbit lengths and velocities. (revolution frequency : number of revolution in the ring per second)

$$f_r = \frac{\beta c}{2\pi R} \quad \rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p} \qquad (1)$$

- Calculate $\frac{d\beta}{\beta}$ as a function of $\frac{dp}{p}$

$$p = \gamma m_0 \beta c \quad \rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\gamma}{\gamma} = (1-\beta^2)^{-1}\frac{d\beta}{\beta} = \gamma^2 \frac{d\beta}{\beta} \qquad (2)$$

- From (1) and (2), we get the relative change in revolution frequency

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{dp}{p} = \eta \frac{dp}{p} \qquad (3)$$

$\eta = \gamma^{-2} - \alpha_c = \gamma^{-2} - \gamma_t^{-2}$ is the slip factor

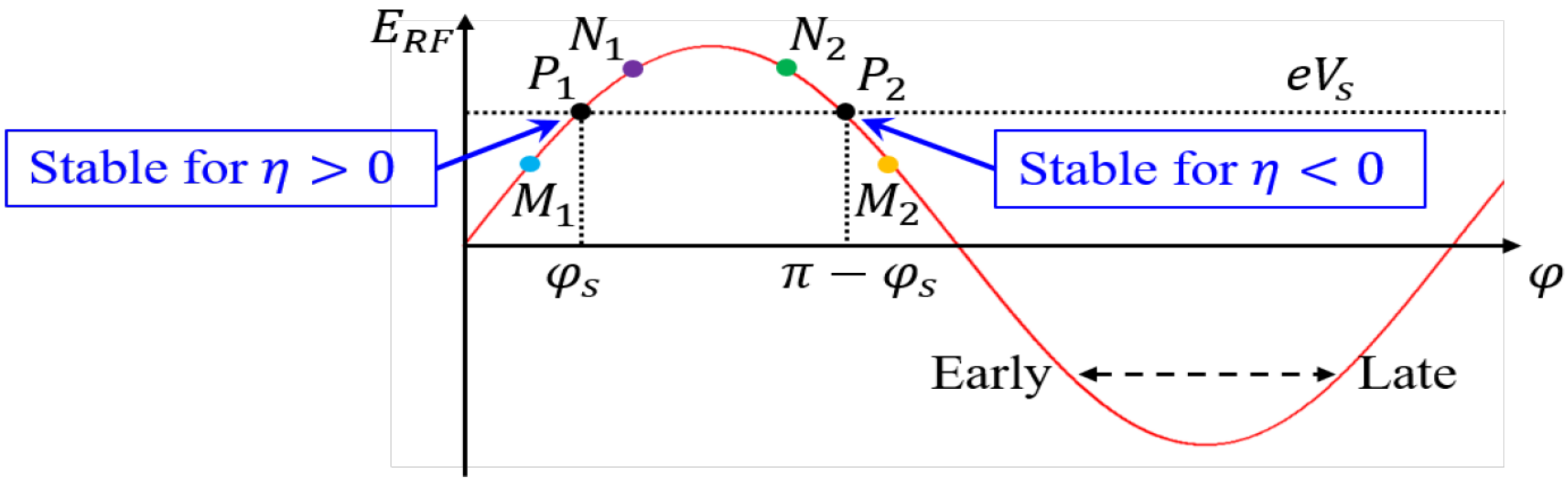
- Transition energy is when $\gamma = \gamma_t = \alpha_c^{-1/2}$ and $\eta = 0$.

Transition Energy

Phase stability in a synchrotron

- Below transition ($\eta > 0 \;\rightarrow\; \gamma < \gamma_{t,}\; \frac{df_r}{f_r} > 0$) gives a higher revolution frequency (increase in velocity dominated)
- Above transition ($\eta < 0 \;\rightarrow\; \gamma > \gamma_{t,}\; \frac{df_r}{f_r} < 0$) gives a lower revolution frequency $\upsilon \approx c$ and a longer path (momentum compaction dominated)

$$\eta = \frac{1}{\gamma^2} - \alpha_c = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$$



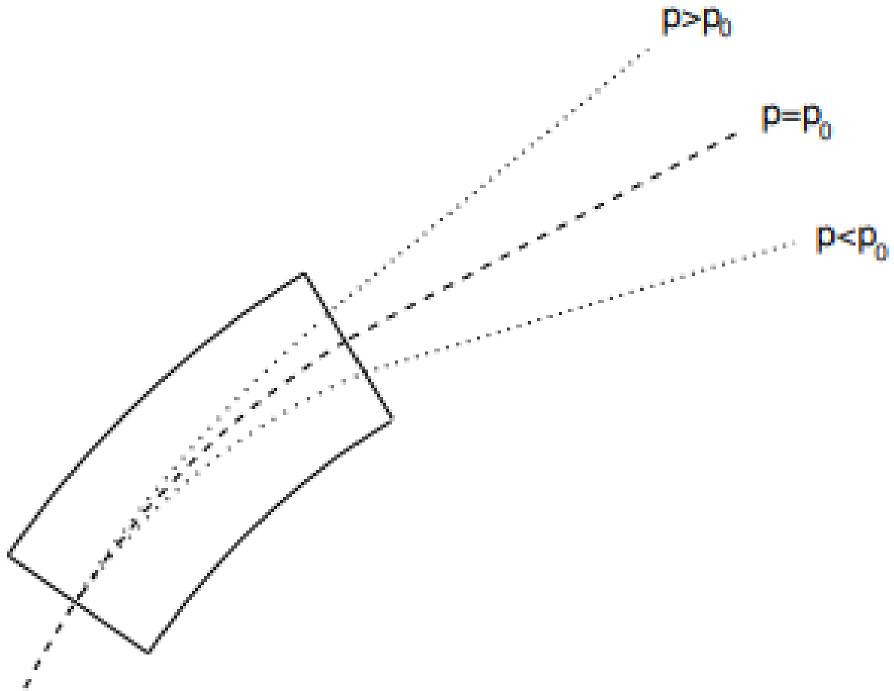
Dispersion function

Beams have small momentum spread among all particles : $\Delta P = P - P_0 \neq 0$.

Dipole introduces a linear correlation between transverse position and momentum, called η(s) :

$$x(s) = \eta(s)\,\frac{\Delta P}{P_0}$$

This correlation is known as **dispersion function.**



Inhomogeneous Hill's equation

If $P \neq P_0$ (define $\delta = \frac{P-P_0}{P_0} = \frac{\Delta P}{P_0}$)

$$B\rho = \frac{P}{q} = \frac{P_0(1+\delta)}{q} = B\rho_0\,(1+\delta) \qquad \Rightarrow \qquad \rho = \rho_0\,(1+\delta\,).$$

When we derived the equation of motion at slide 21:

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{B_0}{\frac{p}{e}} - \frac{xg}{\frac{p}{e}} \quad \text{that became: } x'' + \left(\frac{1}{\rho^2} + k\right) x = 0.$$

$$\frac{B_0}{\frac{p}{e}} = \frac{eB_0}{P_0+\Delta P} \approx \frac{eB_0}{P_0}\left(1 - \frac{\Delta P}{P_0}\right) \approx \frac{1}{\rho}(1-\delta)\,.$$

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{\delta}{\rho}$$

If we drop the suffix 0 , this is " the inhomogeneous Hill's equation" :

$$x'' + \left(\frac{1}{\rho^2} + k\right) x = \frac{1}{\rho}\,\frac{\Delta P}{P_0}$$

Solution of the inhomogeneous Hill's equation

A particle with $\Delta P = P - P_0 \neq 0$ has total deviation of the particle from the reference orbit

$$x(\text{s}) = x_\beta\,(s) + x_\varepsilon(s)$$

- ► $x_\beta(s)$ describes the betatron oscillation around the new closed orbit, and

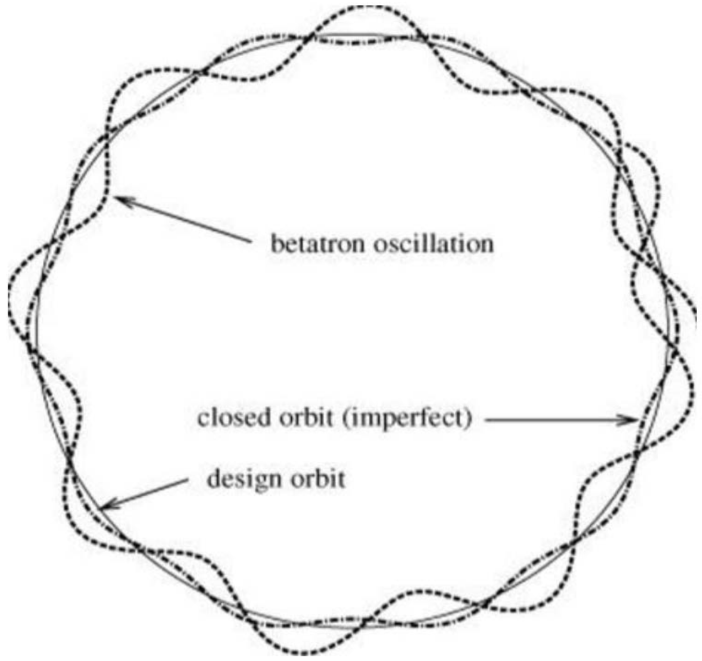
  it's the solution of the homogeneous equation $x''_\beta(\text{s}) + K(s)x_\beta\,(s) = 0$

- ► $x_\varepsilon(s)$ describes the deviation of the closed orbit for an off-momentum particle.

  $x_\varepsilon\,(s) = \eta(s)\,\frac{\Delta P}{P_0}$, where η(s) is the solution of the equation

$$\boldsymbol{\eta''(s) + K(s)\eta(s) = \frac{1}{\rho}}$$

$$x'' + \left(\frac{1}{\rho^2} + k\right)x = \frac{1}{\rho}\frac{\Delta P}{P_0}$$



Dispersion function and orbit

It can be shown that the solution is:

$$\eta(s) = S(s)\int_0^s \frac{1}{\rho(s')} C\,(s')\,ds' \; - C(s)\int_0^s \frac{1}{\rho(s')} S\,(s')\,ds'$$

Once we know $\eta(s)$, the orbit $x(s) = x_\beta\,(s) + x_\varepsilon(s)$, with $x_\varepsilon(s) = \eta(s)\frac{\Delta P}{P_0}$, can be rewritten as

$$x(s) = x_\beta\,(s) + x_\varepsilon(s)$$

$$= C(s)\,x_0 + S(s)x_0' + \eta(s)\,\frac{\Delta P}{P_0}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta P}{P_0}\begin{pmatrix} \eta \\ \eta' \end{pmatrix}_0$$

Dispersion function and orbit

In a more compact way:

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} C & S & \eta \\ C' & S' & \eta' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0 \quad , \quad \eta(s) = S(s)\int_0^s \frac{1}{\rho(s')} C\,(s')\,ds' - C(s)\int_0^s \frac{1}{\rho(s')} S\,(s')\,ds'$$

- Drift space

$M_{Drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$, C(t) = 1, S(t) = L, ρ(t) = ∞ ⇒ the integrals cancel

$$M_{Drift} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_s = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta P/P_0 \end{pmatrix}_0$$

$$\eta = 0 \qquad \eta' = 0$$

Dispersion function in a sector dipole

- **Sector dipole:** $\boldsymbol{K = \frac{1}{\rho^2}}$ **:**

$$M_{Dipole} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,L) \\ -\sqrt{K}\sin(\sqrt{K}\,L) & \cos(\sqrt{K}L) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\,\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

which gives

$$\eta(L) = \rho\left(1-\cos\frac{L}{\rho}\right)$$

$$\eta'(L) = \sin\frac{L}{\rho}$$

therefore

$$M_{Dipole} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\,\sin\frac{L}{\rho} & \rho\left(1-\cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

$\phi = \frac{L}{\rho}$ is the bending angle, L is the length of magnet.

Dispersion function in a quadrupole

► **Focusing quadrupole, $K > 0$:**

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) & 0 \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

► **Defocusing quadrupole, $K < 0$:**

$$M_{QD} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh(\sqrt{K}L) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Periodic dispersion

In a periodic lattice, the dispersion must be also periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we need to have:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} m11 & m12 & m13 \\ m21 & m22 & m23 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

Let's rewrite this in 2 x 2 form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} m11 & m12 \\ m21 & m22 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} m13 \\ m23 \end{pmatrix}$$

$$\begin{pmatrix} 1 - m11 & -m12 \\ -m21 & 1 - m22 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} m13 \\ m23 \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{2(1-cos\mu)} \begin{pmatrix} m13 + m12\, m23 - m22\, m13 \\ m23 + m21\, m13 - m11\, m23 \end{pmatrix}$$

Tune shift 와 Chromaticity

사극자석에 집속오차 (gradient error) 가 있으면 베타트론 진동수가 변한다

작은 집속오차 k 가 국부적으로 존재시 전송행렬

$$M = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} M_0 = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \text{a}\sin\mu_0 \end{pmatrix}$$

$$M = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ k - (\cos\mu_0 + \alpha_0 \sin\mu_0) - \gamma_0 \sin\mu_0 & -k\beta_0 \sin\mu_0 + (\cos\mu_0 - \alpha_0 \sin\mu_0) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

$$\text{Trace}(M) \equiv 2\cos\mu = 2\cos\mu_0 - k\beta_0 \sin\mu_0$$

k가 작으므로 $\mu = \mu_0 + \Delta\mu$ 라고 하면

$$\cos\mu = \cos(\mu_0 + \Delta\mu) = \cos\mu_0 \cos\Delta\mu - \sin\mu_0 \sin\Delta\mu$$

$$\rightarrow \cos\mu_0 - \frac{k}{2}\beta_0 \sin\mu_0$$

$$\Delta\mu = \frac{k}{2}\beta_0 \qquad (\mu = 2\pi\nu)$$

Tune shift : $\Delta\nu = \frac{k}{4\pi}\beta_0$ 일반적으로 $\Delta\nu = \frac{1}{4\pi}\int \Delta k\, \beta(S)\, \mathrm{d}S$ $(k = \Delta k\ \mathrm{l})$

Chromaticity (색수차)

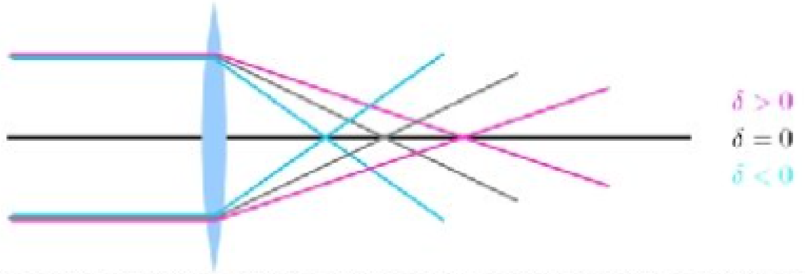
어떤 입자의 운동량이 중심운동량에서 ΔP만큼 벗어났다면
입자가 느끼는 집속력은 $\frac{k}{1+\frac{\Delta p}{p}} \approx k - k\frac{\Delta p}{p}$ 이 되고
중심운동량을 가진 입자보다 집속력이 $k\frac{\Delta p}{p}$ 만큼 약해진다.
→ 이 입자에 대해서 Tune이 $\Delta\nu = -\frac{1}{4\pi}\int k\beta(s)\,\mathrm{d}s\,\frac{\Delta p}{p}$ 만큼 줄어든다.

$$\xi = \frac{\Delta\nu}{\frac{\Delta p}{p}}$$: chromaticity

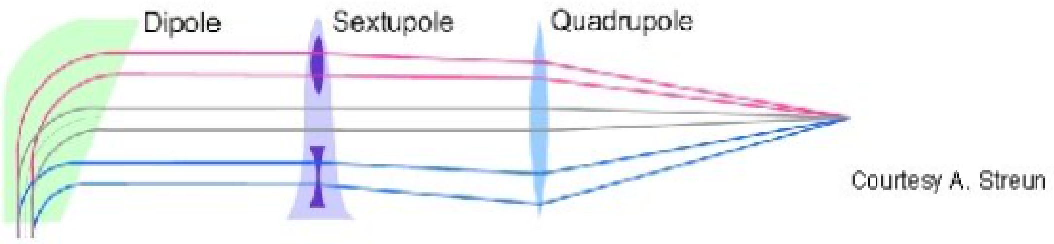
$$\xi = -\frac{1}{4\pi}\int k\beta(s)\,\mathrm{d}s$$

설계 단계에서 존재하며 사극자석에서 β가 크면
ξ의 절대값이 커짐.

Small emittance → Strong quadrupoles → Large (natural) chromaticity

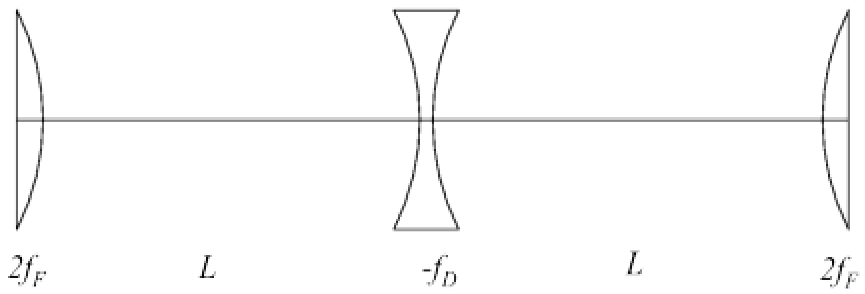


→ strong sextupoles (sextupoles guarantee the focussing of off-energy particles)



Tune shift가 크면 운동이 불안정해지고
β함수도 $\frac{\Delta p}{p}$와 함께 크게 왜곡된다.
Head-Tail instability가 발생 않도록 ξ를 거의
0으로 되게끔 색수차 보정을 육극자석으로 사용 수행한다.

FODO의 베타함수, 색수차 (I)



- 베타함수의 최대는 집속 사극자석 중심점에서 시작
- 길이 L인 drift와 얇은 렌즈 근사치로 전송행렬은

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \quad (1)$$

- β 는 집속 사극자석의 중심점에서 최대값을 갖으므로

$$\alpha = -\frac{\beta'}{2} = 0$$

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \quad (2)$$

FODO의 베타함수, 색수차 (I)

- 식 (1) =식 (2)

$$\cos\mu = \frac{1}{2}tr(M) = 1 + \frac{1}{f_D} - \frac{1}{f_F} - \frac{L^2}{2f_0f_F} = 1\text{-}2\sin^2\frac{\mu}{2}$$

$f_D = f_F$이면, $\sin\frac{\mu}{2} = \frac{L}{2f}$ 이 된다.

Stability 조건은 $f > \frac{L}{2}$

- 전송행렬의 M12 성분

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

$$\beta_{max} = \frac{2L + \frac{L^2}{D}}{\sin\mu} = \frac{2L(1+\sin\frac{\mu}{2})}{\sin\mu}$$

- 최소 베타 함수값은 발산 사극자석의 위치에서 일어나므로 ($f_D = -f_F$)

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} = \frac{2L(1-\sin\frac{\mu}{2})}{\sin\mu}$$

FODO의 베타함수, 색수차 (I)

- natrual 색수차는 사극자석의 에너지 의존성으로부터 유래

$$\xi = -\frac{1}{4\pi}\oint \beta(s)k(s)\,ds$$

$$= -\frac{1}{4\pi}\times N_{Cell}\int_{cell}\beta(s)k(s)\ ds$$

$$= -\frac{N_{Cell}}{4\pi}\left[\beta_{max}\frac{1}{2f_F} + \beta_{min}(-\frac{1}{f_D}) + \beta_{max}\frac{1}{2f_F}\right]$$

$$= -\frac{N_{Cell}}{2\pi}\frac{L}{\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]$$

$f_D = f_F$ 이면

$$\xi = -\frac{N_{Cell}}{2\pi\sin\mu}\frac{L^2}{f2} = -\frac{N_{Cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}}4\sin^2\frac{\mu}{2} \qquad (\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2} \qquad \frac{L^2}{f_0 f_F} = 4\sin^2\frac{\mu}{2})$$

$$= -\frac{N_{Cell}}{\pi}\tan\frac{\mu}{2}$$

FODO의 분산함수 (I)

전송행렬은 $\frac{QF}{2}$,B ,QD ,B, $\frac{QF}{2}$ 에서

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & \frac{l\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l & \frac{l\theta}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{l^2}{2f^2} & 2l(1 + \frac{l}{2f}) & 2l\theta(1 + \frac{l}{4f}) \\ -\frac{l}{2f}(1 - \frac{l^2}{2f^2}) & 1 - \frac{l^2}{2f^2} & 2\theta(1 - \frac{l}{2f} - \frac{l^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix}$$

l, θ는 이극자석 길이, 휨 각도, f는 사극자석의 집속 세기이다.

$$\cos\mu = 1 - 2\sin^2\frac{\mu}{2} = \frac{2}{2}(1 - \frac{l^2}{2f^2})$$

$$\sin\frac{\mu}{2} = \frac{l}{2f}$$

FODO의 분산함수 (I)

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = M' \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = (I\text{-}M') \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{2(1-\cos\mu)} \begin{pmatrix} m_{13} + m_{12}m_{23} - m_{22}m_{13} \\ m_{23} + m_{21}m_{13} - m_{11}m_{23} \end{pmatrix}$$

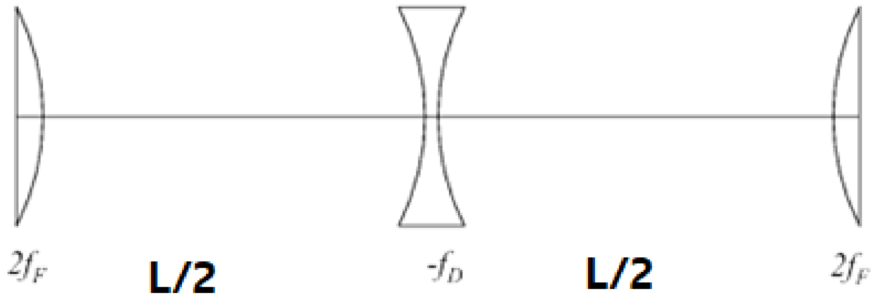
$$\eta_F = \frac{l\theta(1+\frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}} \qquad \eta_{D'_F} = 0$$

$$\eta_D = \frac{l\theta(1-\frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}} \qquad \eta_{D'_D} = 0$$

FODO의 베타함수, 색수차 (II)

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L(1 + \frac{L}{4f}) \\ \frac{L}{4f^2}(\frac{L}{4f} - 1) & 1 - \frac{L^2}{8f^2} \end{pmatrix}$$



- QF 중심에서 β함수가 최대이므로 $\alpha(s) = -\frac{\beta'(s)}{2} = 0$

$$M = \begin{pmatrix} \cos\mu & \beta_{max} \sin\mu \\ -\frac{\sin\mu}{\beta_{max}} & \cos\mu \end{pmatrix}$$

- $\sin\frac{\mu}{2} = \frac{L}{4f}$ stability 조건은 $f > \frac{L}{4}$

$$\beta_{max} = L\left(\frac{1+\sin\frac{\mu}{2}}{\sin\mu}\right)$$

$$\beta_{min} = L\left(\frac{1-\sin\frac{\mu}{2}}{\sin\mu}\right)$$

- 전체 길이가 2L일 경우에 2L → L로 변경하면 된다

FODO의 분산함수, 색수차 (II)

- Dispersion 성분이 포함된 전송행렬 : θ << 1 일차항만 사용

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta}{8} \\ 0 & 1 & \frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} & \frac{L\theta}{8} \\ 0 & 1 & \frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{L^2}{8f^2} & L+\frac{L^2}{4f} & \frac{L}{2}(1+\frac{L}{8f})\theta \\ -\frac{1}{4f^2}(1-\frac{L}{4f}) & 1-\frac{L^2}{8f^2} & (1-\frac{L}{8f}-\frac{L^2}{32f^2})\theta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = M' \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix}$$

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = (1\text{-}M') \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix}$$

$$\eta_{max} = \frac{l\theta}{4}\left(\frac{1+\frac{L}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right) \qquad \eta' = 0$$

$$\eta_{min} = \frac{l\theta}{4}\left(\frac{1-\frac{L}{2}\sin\frac{\mu}{2}}{\sin^2\frac{\mu}{2}}\right) \qquad \eta' = 0$$

FODO의 분산함수, 색수차 (II)

- β_x 값이 크고 β_y 값이 작은 지점에 k_s>0 세기를 갖는 육극자석 설치하여 수직 색수차에 적은 영향을 주면서 수평 색수차 보정
- β_y값이 크고 β_x가 작은 지점에 k_s<0 세기를 갖는 육극자석 설치하여 수평 색수차에 적은 영향을 주면서 수직 색수차를 보정
- 색수차는 $\xi = -\frac{1}{\pi}\tan\frac{\mu}{2}$ 이므로 μ = 90°에 대하여 수평,수직 모두 $\xi = -\frac{1}{\pi}$이 된다.
- 색수차 보정을 위해 육극자석들의 세기 조정

$$-\frac{1}{4\pi}[K_{2F}\ \eta_{max}\ \beta_{max} + K_{2D}\ \eta_{min}\ \beta_{min}] = -\frac{1}{\pi}$$

$$\beta_{max} = \frac{L+\frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f}$$

$$\beta_{min} = \frac{L-\frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f}$$

$$\eta_{max} = \frac{f}{L}(4f+\frac{L}{2})\theta$$

$$\eta_{min} = \frac{f}{L}(4f-\frac{L}{2})\theta$$

$$-\frac{1}{4\pi}\frac{f}{L}\theta[K_{2F}(4f+\frac{L}{2})(L+\frac{L^2}{4f}) + K_{2D}(4f-\frac{L}{2})(L-\frac{L^2}{4f})] = -\frac{1}{\pi}$$

Disturbed closed orbits

S =0 에서 ΔS 구간에 dipole field error가 있을때

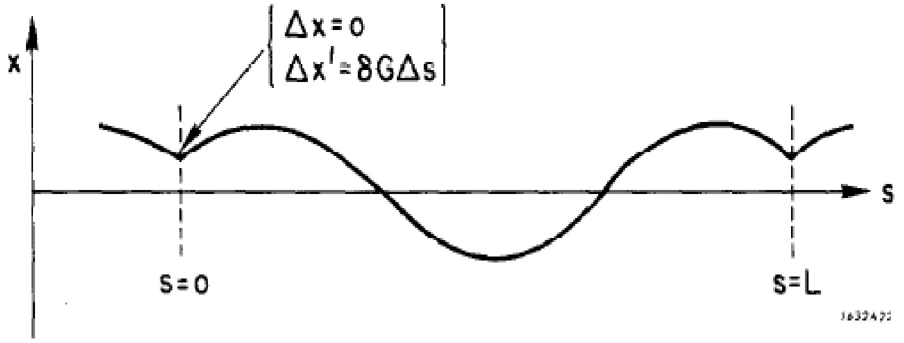
x is unchanged

x' is changed by amount $\Delta x' = \delta G\ \Delta s = \frac{ec\delta B}{E_0}\Delta s$

$$x'' = k(x)x + \delta G(s)$$

$$x' \rightarrow x' + \Delta x' = x' + \delta G \Delta s$$

$$\begin{cases} x(0) = 0 \\ x'(0) = \Delta x' \end{cases}$$



The disturbed closed orbit for a field error at s = 0.

$S = 0$에서 ΔS 구간에 δG에러가 있는 경우

ΔS 이외 구간 $x(s) = a\sqrt{\beta(s)}cos(\phi(S) - \theta),\ s \neq 0$

$$\begin{cases} x_c(L) = x_c(0) & (1) \\ x'_c\ (L) + \delta G\Delta S = x'_c(0) & (2) \end{cases}$$

Disturbed closed orbits

조건 (1)

$$x_c(L) = a\sqrt{\beta(L)}\cos(\phi(L) - \theta)$$
$$= a\sqrt{\beta(0)}\cos(2\pi\nu - \theta)$$

$$x_c(0) = a\sqrt{\beta(0)}\cos(\phi(0) - \theta)$$
$$= a\sqrt{\beta(0)}\cos(\theta)$$

$$\cos(2\pi\nu - \theta) = \cos\theta$$

$$\begin{cases} \cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \\ \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \end{cases}$$

$$\cos(2\pi\nu - \theta) - \cos\theta = 0$$

$$-2\sin\frac{(2\pi\nu-\theta)+\theta}{2}\cdot\sin\frac{(2\pi\nu-\theta)-\theta}{2} = 0$$
$$(-2)\sin\pi\nu\cdot\sin(\pi\nu - \theta) = 0$$

$$\therefore\ \theta = \pi\nu$$

Disturbed closed orbits

조건 (2)

$$x'_c\ (s) = -\frac{a}{\sqrt{\beta(s)}}\sin(\phi(s)-\theta) + \frac{\beta'(s)}{2\beta(s)}x(s)$$

$$x'_c\ (L) = -\frac{a}{\sqrt{\beta(\mathrm{L})}}\sin(\phi(\mathrm{L})-\theta) + \frac{\beta'(\mathrm{L})}{2\beta(\mathrm{L})}x(L)$$

$$= \frac{a}{\sqrt{\beta(0)}}\{-\sin(2\pi\nu-\pi\nu)\} + \frac{\beta'(0)}{2\beta(0)}a\sqrt{\beta(0)}\cos(\pi\nu)$$

$$= \frac{a}{\sqrt{\beta(0)}}\{-\sin(\pi\nu) + \frac{\beta'(0)}{2}\cos(\pi\nu)\}$$

$$\equiv x'(0) - \delta G\Delta S$$

$$= -\frac{a}{\sqrt{\beta(0)}}\sin(-\theta) + \frac{\beta'(0)}{2\beta(0)}x(0) - \delta G\Delta S$$

$$= -\frac{a}{\sqrt{\beta(0)}}\sin(\theta) + \frac{\beta'(0)}{2\beta(0)}a\sqrt{\beta(0)}\cos(\theta) - \delta G\Delta S$$

$$= -\frac{a}{\sqrt{\beta(0)}}(\sin\theta + \frac{1}{2}\beta'(0)\cos(\theta)) - \delta G\Delta S$$

$$\therefore -\frac{2a}{\sqrt{\beta(0)}}\sin\pi\nu = -\delta G\Delta S$$

$$a = \frac{\sqrt{\beta(0)}\delta G\Delta S}{2\sin\pi\nu}$$

Disturbed closed orbits

Distorted closed orbit induced by field error δG at s = 0 ~ ΔS is

$$x_c(s) = \frac{\delta G \Delta S \sqrt{\beta(0)}}{2\sin \pi\nu} \sqrt{\beta(s)} \cos\{\phi(s) - \pi\nu\}$$

$\nu \longrightarrow$ integer, $x(s) \to \infty$ integer resonance

$s = 0$ 에서의 진폭

$$x(0) = \frac{\delta G \Delta S \sqrt{\beta(0)}}{2\sin \pi\nu} \sqrt{\beta(0)} \cos\{\phi(0) - \pi\nu\}$$

$$= \frac{\delta G \Delta S\, \beta(0)}{2\tan \pi\nu}$$

$s = s_1$ 에서 에러가 있으면

$$x(s_1) = \frac{\delta G \Delta S\, \beta(s_1)}{2\tan \pi\nu}$$

저장링의 임의의 곳에 에러가 있다면 $(\bar{S})$

$$s = 0 \to \bar{s}$$

$$\phi(s) \to \phi(s) - \phi(\bar{s})$$

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin \pi\nu} \oint \delta G(\bar{s}) \sqrt{\beta(\bar{s})} \cos\{\phi(s) - \phi(\bar{s}) - \pi\nu\}\, d\bar{s}$$

사극자석 세기 에러 및 베타함수

사극자석 세기 에러는 베타트론 튠 및 베타함수에 영향을 미친다.
저장링을 2개의 부분으로 각각의 전송행렬을 A및 B로 표시한다.

$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

A와 B 사이에 사극자석 에러를 위한 전송행렬

$$M_{dist} = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta kds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta kdsa_{11} + a_{12} & -\Delta kdsa_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & , b_{11}\, a_{12} + b_{12}(-\Delta kdsa_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$