HW4

April 7, 2026

Problem 1

$$\begin{aligned}\mathbf{M} &= \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \\ &= \mathbf{I}\cos\mu + \mathbf{J}\sin\mu \end{aligned} \tag{1}$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$.

1) Show that $\mathbf{M}^2 = \mathbf{I}\cos(2\mu) + \mathbf{J}\sin(2\mu)$
2) Show that $\mathbf{M}^n = \mathbf{I}\cos(n\mu) + \mathbf{J}\sin(n\mu)$ for a positive integer n.

Problem 2

From the definition of the transfer matrix

$$\mathbf{M} = \begin{pmatrix} \cos\mu_x + \alpha_x\sin\mu_x & \beta_x\sin\mu_x \\ -\gamma\sin\mu_x & \cos\mu_x - \alpha_x\sin\mu_x \end{pmatrix} = \mathbf{I}\cos\mu_x + \mathbf{J}\sin\mu_x, \tag{2}$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}$, show that

$$\beta_x\gamma_x - \alpha_x^2 = 1 \tag{3}$$

Hint) use the symplectic condition of $\mathbf{M}$

Problem 3

Using the invariant

$$2J_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \tag{4}$$

show that Courant-Snyder parameter transform from s_0 to s_1 by the matrix transformation

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} \tag{5}$$

if

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{6}$$

Problem 4

With the normalization of x by $\sqrt{\beta_x(s)}$,

$$\eta_x(s) = \frac{x(s)}{\sqrt{\beta_x(s)}} \tag{7}$$

show that

$$\frac{d\eta_x}{d\phi_x} = \frac{\alpha_x x + \beta_x x'}{\sqrt{\beta_x}} \nu_x \tag{8}$$

and

$$\frac{d^2\eta_x}{d\phi_x^2} + \nu_x^2 \eta_x = 0. \tag{9}$$

Hint) use relationships

$$\begin{aligned} \alpha_x' &= \beta_x K_x - \gamma_x, \\ \beta_x' &= -2\alpha_x, \\ \gamma_x' &= 2\alpha_x K_x, \end{aligned} \tag{10}$$

and Hill's equation.