4. Linear Transverse Motions

Chong Shik Park, Ph.D.

April 7, 2026

[1]:
```
%%latex
\tableofcontents
```

Contents

1 Transverse Motion

This section will cover an introduction to the linear transverse motion in accelerators.

1.1 Cyclotron Motion and Magnetic Rigidity

The force acting on a charged particle in a uniform magnetic field, $\vec{B} = (0, 0, B_0)$, is given by

$$\vec{F} = q(\vec{v} \times \vec{B}) = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m_0 v) \tag{1}$$

In a relativistic regime, m can be represented as γm_0. When the coordinate systems is changed into cylindrical coordinates, (ρ, θ, z),

$$\begin{cases} F_\rho &= \frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\theta}^2 = -q\rho\dot{\theta}B_0 \\ F_\theta &= \frac{1}{\rho}\frac{d}{dt}(m\rho^2\dot{\theta}) = q\dot{\rho}B_0 \\ F_z &= \frac{d}{dz}(m\dot{z}) = 0 \end{cases} \tag{2}$$

If $\rho = \rho_0 = \text{const}$, i.e., $\dot{\rho} = 0$, then

$$\dot{\theta} = \frac{q}{m}B_0 = \omega_c \tag{3}$$

where ω_c is the **cyclotron frequency**. With a small angle deviation in the vertical direction, the particle with the cyclotron motion will spiral up or down.

By replacing $\dot{\theta}$ to v_0/ρ, which is an angular velocity,

$$\frac{mv_0}{q} = B\rho \tag{4}$$

then,

$$(B\rho) = \frac{p}{q} = (10^9/c)p[GeV/c] \approx 3.3356p[GeV/c] \tag{5}$$

This is called the **magnetic rigidity**. A particle with higher momentum will have a higher resistance to deflection by a magnetic field.

1.2 Curvilinear Coordinate System Revisit

In the **Frenet-Serret** coordinate system, the independent variable is the location of the particle on the reference orbit, s. Then the three unit vectors $\hat{s}, \hat{x}, \hat{y}$ are defined locally, and vary with s:

$$\hat{y}(s) = \hat{s} \times \hat{x}(s) \tag{6}$$

$$\frac{d\hat{s}}{ds} = -\frac{\hat{x}}{\rho(s)}, \quad \frac{d\hat{x}}{ds} = \frac{\hat{s}}{\rho(s)}, \quad \frac{d\hat{y}}{ds} = 0 \tag{7}$$

where $\rho(s)$ is the radius of the curvature of the reference orbit.

The particle trajectory around the reference orbit can be expressed as

$$\vec{r}(s) = \vec{r}_0(s) + x(s)\hat{x} + y(s)\hat{y} \tag{8}$$

And the Lame parameters for the Frenet-Serret coordinate system are defined as

$$h_s = 1 + \frac{x}{\rho}, \quad h_x = h_y = 1 \tag{9}$$

1.3 Hamiltonian of Frenet-Serret Coordinate System

In Frenet-Serret Coordinate system, the new Hamilton becomes

$$H = q\phi + \left[(mc^2)^2 + \frac{(p_s - qA_s)^2}{(1 + x/\rho)^2} + (p_x - qA_x)^2 + (p_y - qA_y)^2\right]^{1/2} \tag{10}$$

when $(\phi, \vec{A})$ are scalar and vector potentials.

Then, Hamilton's equation becomes

$$\begin{aligned} \dot{s} &= \frac{\partial H}{\partial p_s}, & \dot{p}_s &= -\frac{\partial H}{\partial s}, \\ \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p}_x &= -\frac{\partial H}{\partial x}, \\ \dot{y} &= \frac{\partial H}{\partial p_y}, & \dot{p}_y &= -\frac{\partial H}{\partial y}. \end{aligned}$$

Now, we define new conjugate phase space coordinates $(x, p_x, y, p_y, t, -H)$ with s as a independent coordinate variable, then

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\dot{x}}{\dot{s}} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = -\frac{\partial p_s}{\partial p_x},$$

$$p_x' = \frac{dp_x}{ds} = \frac{dp_x}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\dot{p_x}}{\dot{s}} = -\left(\frac{\partial H}{\partial x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = \frac{\partial p_s}{\partial x},$$

Similarly,

$$\begin{aligned} y' &= -\frac{\partial p_s}{\partial p_y}, \\ p_y' &= \frac{\partial p_s}{\partial y}, \\ t' &= -\frac{\partial p_s}{\partial(-H)} = \frac{\partial p_s}{\partial H}, \\ H' &= \frac{\partial p_s}{\partial t} \end{aligned} \tag{11}$$

and the new Hamiltonian, $\tilde{H}$, with the new conjugate coordinates becomes

$$\begin{aligned} \tilde{H} &= \tilde{H}(x, p_x, y, p_y, t, -H) = -p_s \\ &= -qA_s - \left(1 + \frac{x}{\rho}\right) \left[\frac{(H - q\phi)^2}{c^2} - (mc^2)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2\right]^{1/2} \end{aligned} \tag{12}$$

1.4 Magnetic Fields in Accelerators

Most of magnets in circular accelerators are electrostatic, i.e., the magnetic fields are constant in time.

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \times \vec{B} = 0 \tag{13}$$

Then these equations are satisfied by a field $\vec{B} = (B_x, B_y, B_z)$ with constant B_z, and B_x, B_y given by

$$B_y + iB_x = C_n(x + iy)^n \tag{14}$$

when we assume $B_z = 0$ in multipole magnets.

In general,

$$B_y + iB_x = \sum_{n=0}^{\infty} C_n(x + iy)^n = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{15}$$

is called **Bethe representation**.

1.5 Hamiltonian Equations of Betatron Motion and Hill's Equations

The transverse components of magnetic fields in circular accelerators can be expanded as

$$B_y + iB_x = \mp \frac{(B_0\rho)}{\rho} \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n \tag{16}$$

where $-$ and $+$ signs are used for positive and negative charges, respectively, and

$$\begin{cases} b_n = \frac{1}{B_0 n!} \frac{\partial^n B_y}{\partial x^n} \\ a_n = \frac{1}{B_0 n!} \frac{\partial^n B_x}{\partial y^n} \quad \text{for} \;\; n = 1, \ldots, \infty \end{cases} \tag{17}$$

For example, with a positively charged particle, if $n = 0$(dipole),

$$B_y + iB_x = -B_0, \tag{18}$$

and if $n = 1$(quadrupole),

$$B_y + iB_x = -\frac{(B\rho)}{\rho}(b_1 + ia_1)(x + iy). \tag{19}$$

With $b_1 = -\frac{1}{B_0\rho}\frac{\partial B_y}{\partial x}$ and $a_1 = 0$,

$$\begin{aligned} B_x &= \frac{\partial B_y}{\partial x} y \\ B_y &= \frac{\partial B_y}{\partial x} x \end{aligned} \tag{20}$$

Using Hamilton's equations,

$$x' = \frac{\partial \tilde{H}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{H}}{\partial x}, \quad y' = \frac{\partial \tilde{H}}{\partial p_y}, \quad p_y' = -\frac{\partial \tilde{H}}{\partial y}, \tag{21}$$

and

$$\begin{aligned} B_x &= -\frac{1}{h_s}\frac{\partial A_s}{\partial y} = -\frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial y}, \\ B_y &= \frac{1}{h_s}\frac{\partial A_s}{\partial x} = \frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial x}, \end{aligned} \tag{22}$$

The equations of linear betatron motion are given by

$$\begin{aligned} x'' - \frac{\rho + x}{\rho^2} &= \pm\frac{B_y}{(B\rho)}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 \\ y'' &= \mp\frac{B_x}{(B\rho)}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 \end{aligned} \tag{23}$$

where $p_0 = qB\rho$ is the momentum of the reference particle, and p is the momentum of the particle. These equations can be simplified as

$$x'' + K_x(s)x = 0,$$

$$y'' + K_y(s)y = 0$$

and is called **Hill's Equations**, where

$$\begin{aligned} K_x(s) &= \frac{1}{\rho^2} \mp K_1(s), \\ K_y(s) &= \pm K_1(s), \quad \text{and} \\ K_1(s) &= \frac{B_1(s)}{(B\rho)} = \frac{1}{(B\rho)}\frac{\partial B_x}{\partial y}. \end{aligned} \tag{24}$$

- In a horizontal bending dipole, $K_x = 1/\rho^2$, and $K_y = 0$. - In quadrupole, $1/\rho = 0$, i.e., $K_x = -K_y$. - K_x and K_y are periodic functions of s. i.e., $K_{x,y}(s + L) = K_{x,y}(s)$.

1.6 Hamiltonian Equations of Betatron Motion and Hill's Equations

The transverse magnetic field in a circular accelerator can be expanded in complex form as

$$B_y + iB_x = \mp B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x + iy)^n, \tag{25}$$

where the upper sign convention depends on the charge sign, and B_0 is the reference dipole field. The normal and skew multipole coefficients are defined by

$$\begin{cases} b_n = \dfrac{1}{B_0\, n!} \dfrac{\partial^n B_y}{\partial x^n}, \\ a_n = \dfrac{1}{B_0\, n!} \dfrac{\partial^n B_x}{\partial x^n}, \end{cases} \qquad n = 0, 1, 2, ... \tag{26}$$

where the derivatives are evaluated on the reference orbit.

For example, for a positively charged particle:

1.6.1 Dipole ($n = 0$)

$$B_y + iB_x = -B_0, \tag{27}$$

so that

$$B_y = -B_0, \qquad B_x = 0. \tag{28}$$

1.6.2 Quadrupole ($n = 1$)

$$B_y + iB_x = -B_0(b_1 + ia_1)(x + iy). \tag{29}$$

For a normal quadrupole, $a_1 = 0$, so

$$B_y + iB_x = -B_0 b_1 (x + iy). \tag{30}$$

Thus,

$$B_y = -B_0 b_1 x, \qquad B_x = -B_0 b_1 y. \tag{31}$$

If we define the quadrupole gradient by

$$B_1(s) \equiv \frac{\partial B_y}{\partial x}, \tag{32}$$

then for the above convention,

$$b_1 = -\frac{1}{B_0} \frac{\partial B_y}{\partial x} = -\frac{B_1}{B_0}, \tag{33}$$

and therefore

$$B_y = B_1 x, \qquad B_x = B_1 y. \tag{34}$$

Using Hamilton's equations,

$$x' = \frac{\partial \tilde{H}}{\partial p_x}, \qquad p_x' = -\frac{\partial \tilde{H}}{\partial x}, \qquad y' = \frac{\partial \tilde{H}}{\partial p_y}, \qquad p_y' = -\frac{\partial \tilde{H}}{\partial y}, \tag{35}$$

together with the relations between the magnetic field and the longitudinal vector potential A_s,

$$\begin{aligned} B_x &= -\frac{1}{h_s}\frac{\partial A_s}{\partial y} = -\frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial y}, \\ B_y &= \frac{1}{h_s}\frac{\partial A_s}{\partial x} = \frac{1}{1 + x/\rho}\frac{\partial A_s}{\partial x}, \end{aligned} \tag{36}$$

the transverse equations of motion can be written as

$$\begin{aligned} x'' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) &= \pm\frac{B_y}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2, \\ y'' &= \mp\frac{B_x}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2, \end{aligned} \tag{37}$$

where

$$p_0 = q(B\rho) \tag{38}$$

is the momentum of the reference particle, and p is the particle momentum.

For small transverse displacements $|x| \ll \rho$, and for $p \approx p_0$, the equations can be linearized. Using

$$B_y \approx -B_0 + B_1(s)x, \qquad B_x \approx B_1(s)y, \tag{39}$$

we obtain

$$x'' + K_x(s)x = 0, \qquad y'' + K_y(s)y = 0, \tag{40}$$

which are called **Hill's equations**, where

$$\begin{aligned} K_x(s) &= \frac{1}{\rho^2} - K_1(s), \\ K_y(s) &= K_1(s), \\ K_1(s) &= \frac{B_1(s)}{B\rho}. \end{aligned} \tag{41}$$

Here, $K_1(s)$ is the normalized quadrupole focusing strength.

1.6.3 Remarks

- In a horizontal bending dipole with no quadrupole gradient, $K_1 = 0$, so

$$K_x = \frac{1}{\rho^2}, \qquad K_y = 0. \tag{42}$$

- In a pure quadrupole, $1/\rho = 0$, so

$$K_x = -K_1, \qquad K_y = K_1, \tag{43}$$

  hence $K_x = -K_y$.

- In a circular accelerator, $K_x(s)$ and $K_y(s)$ are periodic functions of s:

$$K_x(s+L) = K_x(s), \qquad K_y(s+L) = K_y(s), \tag{44}$$

  where L is the circumference or lattice period.

1.7 Transfer Matrix

Starting from Hill's equation:

$$x'' + K_x(s)x = 0, \tag{45}$$

with $K_x(s) = K_x(s+L)$, its solution with constant K_x is given by,

$$x = \begin{cases} A\cos(\sqrt{K_x}s + B) & \text{for} \quad K_x > 0, \\ As + B & \text{for} \quad K_x = 0, \\ A\cosh(\sqrt{|K_x|}s + B) & \text{for} \quad K_x < 0. \end{cases} \tag{46}$$

then,

$$x' = \begin{cases} -A\sqrt{K_x}\sin(\sqrt{K_x}s + B) & \text{for} \quad K_x > 0, \\ A & \text{for} \quad K_x = 0, \\ A\sqrt{|K_x|}\sinh(\sqrt{|K_x|}s + B) & \text{for} \quad K_x < 0. \end{cases} \tag{47}$$

where integration constants A and B can be determined by initial conditions.

If a particle's coordinate at s is defined as $\vec{x}(s) = (x(s), x'(s))$, then the particle's transfer from s_0 to s is given by

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \vec{x}(s) = \mathbf{M}(s|s_0)\vec{x}(s_0) = \mathbf{M}(s|s_0)\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \tag{48}$$

where $\mathbf{M}$ is the betatron transfer matrix.

From the linear solutions of Hill's equations, the transfer matrix for a constant focusing function K is

$$\mathbf{M}(s|s_0) = \begin{cases} \begin{pmatrix} \cos(\sqrt{K_x}l) & \frac{1}{\sqrt{K_x}}\sin(\sqrt{K_x}l) \\ -\sqrt{K_x}\sin(\sqrt{K_x}l) & \cos(\sqrt{K_x}l) \end{pmatrix} & \text{for} \quad K_x > 0, \\ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} & \text{for} \quad K_x = 0, \\ \begin{pmatrix} \cosh(\sqrt{|K_x|}l) & \frac{1}{\sqrt{|K_x|}}\sin(\sqrt{|K_x|}l) \\ \sqrt{|K_x|}\sinh(\sqrt{|K_x|}l) & \cosh(\sqrt{|K_x|}l) \end{pmatrix} & \text{for} \quad K_x < 0 \end{cases} \tag{49}$$

1.8 Transfer Map

In accelerators, a chain of accelerator elements forms a beam line, which is called a **lattice**.

The transfer matrix of the FODO cell starting with the focusing quadrupole is $\mathbf{M}_{\text{FODO}}$ can be expressed as

$$\mathbf{M}_{\text{FODO}} = \mathbf{M}_{\text{Drift2}}\mathbf{M}_{\text{DQuad}}\mathbf{M}_{\text{Drift1}}\mathbf{M}_{\text{FQuad}}. \tag{50}$$

When the lattice has bending magnets and the sum of bending angles from them is 360 degree, then the lattice makes a circular ring. For example, a FODO lattice including two dipoles with the bending angle, θ, and the FODO cell is repeating N-times to form a ring, then

$$N \times 2\theta = 2\pi, \tag{51}$$

and

$$\theta = \frac{\pi}{N}. \tag{52}$$

The transfer matrix of the FODO cell with two bending magnets is

$$\mathbf{M}_{\text{FODO}} = \mathbf{M}_{\text{Drift4}}\mathbf{M}_{\text{DQuad}}\mathbf{M}_{\text{Drift3}}\mathbf{M}_{\text{Dipole}}\mathbf{M}_{\text{Drift2}}\mathbf{M}_{\text{Dipole}}\mathbf{M}_{\text{Drift1}}\mathbf{M}_{\text{FQuad}}. \tag{53}$$

Therefore, transfer matrix of the ring with the N-FODO cells becomes

$$\begin{aligned}\mathbf{M}_{\text{Ring}} &= \mathbf{M}_{\text{FODO}}\mathbf{M}_{\text{FODO}}\cdots\mathbf{M}_{\text{FODO}}\\ &= \mathbf{M}_{\text{FODO}}^{N},\end{aligned} \tag{54}$$

and is also called one-turn matrix.

1.9 Stability Condition

For n periodic transfer with $\mathbf{M}(s|s_0)$, all elements of transfer matrix, $\mathbf{M}^n = \mathbf{M}\mathbf{M}\cdots\mathbf{M}$ should remain bounded.

Let λ_1 and λ_2 are eigen values of the matrix $\mathbf{M}$, then λ_1 and λ_2 satisfy following relations:

$$\begin{aligned}&\det(\mathbf{M}) = 1 \Rightarrow \lambda_1 = \lambda_2\\ &\lambda_1 + \lambda_2 = \text{Trace}(\mathbf{M})\end{aligned} \tag{55}$$

therefore

$$\lambda^2 - \text{Trace}(\mathbf{M})\lambda + 1 = 0 \tag{56}$$

Let $\text{Trace}(\mathbf{M}) = 2\cos\mu$, then

$$\lambda^2 - 2\lambda\cos\mu + 1 = 0 \tag{57}$$

and if $2\cos\mu \leq 2$, μ is real, however, if $2\cos\mu > 2$, μ is complex.

The eigen values can be written as

$$\lambda_1 = \exp{(i\mu)}, \quad \text{and} \quad \lambda_2 = \exp{(-i\mu)} \tag{58}$$

then μ is the **betatron phase advance**.

1.10 Symplectic Condition

A transfer matrix,

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{59}$$

is **symplectic**, if it satisfies:

$$\mathbf{M}^T \mathbf{S} \mathbf{M} = \mathbf{S}, \tag{60}$$

where $\mathbf{S}$ is the anti-symmetric matrix:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{61}$$

Then,

$$\mathbf{M}^T \mathbf{S} \mathbf{M} = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} 0 & \det(\mathbf{M}) \\ -\det(\mathbf{M}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{62}$$

Therefore, if $\mathbf{M}$ is symplectic, then $\det(\mathbf{M}) = 1$, (or vice versa for 1-D case).

1.11 Courant-Snyder Parameterization (Twiss Parameters)

With the symplectic condition, we can define a transfer matrix in the form:

$$\mathbf{M} = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} = \mathbf{I}\cos\mu_x + \mathbf{J}\sin\mu_x \tag{63}$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{J} = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}$ with $\alpha_x^2 - \beta_x\gamma_x = -1$.

One can easily find that

$$\mathrm{Trace}(\mathbf{J}) = 0 \tag{64}$$

and

$$\mathbf{J}^2 = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix} \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix} = \begin{pmatrix} \alpha_x^2 - \beta_x\gamma_x & 0 \\ 0 & \alpha_x^2 - \beta_x\gamma_x \end{pmatrix} = -\mathbf{I} \tag{65}$$

The transfer matrix $\mathbf{M}$ also satisfies that

$$\mathbf{M}^n = (\mathbf{I}\cos\mu_x + \mathbf{J}\sin\mu_x)^n = \mathbf{I}\cos(n\mu_x) + \mathbf{J}\sin(n\mu_x). \tag{66}$$

1.12 Action-Angle Variables

Assume that we consider only the uncoupled horizontal motion, and introduce the **action variable**, J_x,

$$J_x = \frac{1}{2} \begin{pmatrix} x & x' \end{pmatrix} \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \frac{1}{2} \left(\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2\right). \tag{67}$$

With the symplectic condition, the quantity, J_x, is invariant under the transport along the beamline, even though the values of the phase space coordinates, (x, x'), and Courant-Snyder parameters, α_x, β_x, and γ_x all vary along the beamline.

The above equation for fixed J_x implies an ellipse in phase space. The area of the ellipse is $2\pi J_x$, and the shape is determined by α_x, β_x, and γ_x.

Furthermore, we define the **angle variable**, ϕ_x as follows:

$$\tan\phi_x = -\beta_x \frac{x'}{x} - \alpha_x. \tag{68}$$

Then, the solution of Hill's equation,

$$x'' + K_x(s)x = 0, \tag{69}$$

can be written as

$$x(s) = A\sqrt{\beta_x(s)}\cos\left[\phi_x(s) + \phi_{x0}\right] = \sqrt{2\beta_x J_x}\cos\left[\phi_x(s) + \phi_{x0}\right] \tag{70}$$

Then,

$$\begin{aligned} x'(s) &= -\sqrt{\frac{2J_x}{\beta_x}}\frac{\beta_x'}{2}\sin\left[\phi_x(s)+\phi_{x0}\right] - \sqrt{2\beta_x J_x}\phi_x'\sin\left[\phi_x(s)+\phi_{x0}\right] \\ &= -\sqrt{\frac{2J_x}{\beta_x}}\left\{\sin\left[\phi_x(s)+\phi_{x0}\right] + \alpha_x\cos\left[\phi_x(s)+\phi_{x0}\right]\right\}, \end{aligned} \tag{71}$$

where

$$\begin{aligned} \phi_x &= \int_0^s \frac{ds}{\beta_x}, \\ \phi_x' &= \frac{d\phi_x}{\beta_x} = \frac{1}{\beta_x}, \text{and} \\ \alpha_x &= -\frac{\beta_x'}{2}. \end{aligned} \tag{72}$$

The new Hamiltonian with the **canonical action-angle variables**, (J_x, ϕ_x) is

$$H_x = \frac{J_x}{\beta_x}, \tag{73}$$

with the equations of motion

$$\begin{aligned} \frac{dJ_x}{ds} &= -\frac{\partial H_x}{\partial \phi_x} = 0, \\ \frac{d\phi_x}{ds} &= \frac{\partial H_x}{\partial J_x} = \frac{1}{\beta_x}. \end{aligned} \tag{74}$$

Define new normalized coordinates, (X, X'),

$$\begin{aligned} X &= x = \sqrt{2\beta_x J_x}\cos\mu_x, \\ X' &= \alpha_x x + \beta_x x' = -\sqrt{2\beta_x J_x}\sin\mu_x. \end{aligned} \tag{75}$$

where $\mu_x(s) = \phi_x(s) + \phi_{x0}$ is the **betatron phase advance**.

These new coordinates satisfy that

$$X^2 + X'^2 = \left(\sqrt{2\beta_x J_x}\right)^2 = 2\beta_x J_x. \tag{76}$$

The shape of the normalized phase space is a circle with the radius of $\sqrt{2\beta_x J_x}$and is independent of s.

Furthermore, let's rewrite the solution in the matrix form as

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix} \begin{pmatrix} \sqrt{2J_x}\cos\mu_x \\ -\sqrt{2J_x}\sin\mu_x \end{pmatrix} = \mathbf{B}(s)\tilde{\mathbf{X}}(s), \tag{77}$$

where $\mathbf{B}(s) = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix}$ and $\tilde{\mathbf{X}}(s) = \begin{pmatrix} \sqrt{2J_x}\cos\mu_x \\ -\sqrt{2J_x}\sin\mu_x \end{pmatrix}$.

When a particle is transferred from s_0 to s with a symplectic transfer matrix, $\mathbf{M}(s|s_0)$, then

$$\begin{aligned} \mathbf{X}(s) = \mathbf{M}(s|s_0)\mathbf{X}(s_0) &\Rightarrow \mathbf{B}(s)\tilde{\mathbf{X}}(s) = \mathbf{M}(s|s_0)\mathbf{B}(s_0)\tilde{\mathbf{X}}(s_0) \\ &\Rightarrow \tilde{\mathbf{X}}(s) = \mathbf{B}^{-1}(s)\mathbf{M}(s|s_0)\mathbf{B}(s_0)\tilde{\mathbf{X}}(s_0) \\ &\Rightarrow \tilde{\mathbf{X}}(s) = \tilde{\mathbf{M}}(s|s_0)\tilde{\mathbf{X}}(s_0). \end{aligned} \tag{78}$$

Since the new transfer matrix, $\tilde{\mathbf{M}}(s|s_0)$, is just a rotational transformation matrix with the angle of $\Delta\mu_x = \mu_x(s) - \mu_x(s_0)$,

$$\begin{aligned} \mathbf{M}(s|s_0) &= \mathbf{B}^{-1}(s)\tilde{\mathbf{M}}(s|s_0)\mathbf{B}(s_0) \\ &= \mathbf{B}^{-1}(s)\begin{pmatrix} \cos\Delta\mu_x & \sin\Delta\mu_x \\ -\sin\Delta\mu_x & \cos\Delta\mu_x \end{pmatrix}\mathbf{B}(s_0) \\ &= \begin{pmatrix} \sqrt{\frac{\beta_{x1}}{\beta_{x0}}}(\cos\Delta\mu_x + \alpha_{x0}\sin\Delta\mu_x) & \sqrt{\beta_{x0}\beta_{x1}}\sin\Delta\mu_x \\ \frac{\alpha_{x0}-\alpha_{x1}}{\sqrt{\beta_{x0}\beta_{x1}}}\cos\Delta\mu_x - \frac{1+\alpha_{x0}\alpha_{x1}}{\sqrt{\beta_{x0}\beta_{x1}}}\sin\Delta\mu_x & \sqrt{\frac{\beta_{x0}}{\beta_{x1}}}(\cos\Delta\mu_x - \alpha_{x1}\sin\Delta\mu_x) \end{pmatrix} \end{aligned} \tag{79}$$

1.13 Betatron Tune

Consider a ring of circumference, $C = PL$, with P identical cells of length L. If the phase advance per cell is Φ_x, then the phase change per revolution is $P\Phi_x$. The **betatron tune** is defined as the number betatron oscillation per one revolution,

$$\nu_x(= Q_x) = \frac{P\Phi_x}{2\pi} = \frac{1}{2\pi}P\int_s^{s+L}\frac{ds}{\beta_x(s)} = \frac{1}{2\pi}\int_s^{s+C}\frac{ds}{\beta_x(s)} \tag{80}$$

Normalize x by the betatron function, then

$$\begin{aligned} &\eta_x(s) = \frac{x(s)}{\sqrt{\beta_x(s)}} \\ &\Rightarrow \frac{d\eta_x}{d\phi_x} = \frac{d\eta_x}{ds}\frac{ds}{d\phi_x} = \frac{d}{ds}\left[\frac{x(s)}{\sqrt{\beta_x(s)}}\right]\nu_x\beta_x = \frac{\alpha_x x + \beta_x x'}{\sqrt{\beta_x}}\nu_x \end{aligned} \tag{81}$$

Therefore, the **linear betatron motion**,

$$\frac{d^2\eta_x}{d\phi_x^2} + \nu_x^2\eta_x = 0, \tag{82}$$

becomes a **simple harmonic oscillator**.

1.14 Courant-Snyder Invariant and Emittance

Now, we can construct a general solution of the Hill's equation as

$$x = \sqrt{2\beta_x J_x}\cos\left(\nu_x\phi_x + \phi_{x0}\right) \tag{83}$$

Then,

$$x' = -\sqrt{\frac{2J_x}{\beta_x}}\left[\sin\left(\nu_x\phi_x + \phi_{x0}\right) + \alpha_x\cos\left(\nu_x\phi_x + \phi_{x0}\right)\right] \tag{84}$$

and

$$\alpha_x x + \beta_x x' = -\sqrt{2\beta_x J_x}\sin\left(\nu_x\phi_x + \phi_{x0}\right). \tag{85}$$

This becomes

$$x^2 + (\alpha_x x + \beta_x x')^2 = 2\beta_x J_x, \tag{86}$$

or

$$2J_x = \frac{1}{\beta_x}\left[x^2 + (\alpha_x x + \beta_x x')^2\right] = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2, \tag{87}$$

which is invariant, i.e., independent of s. Therefore, the trajectory of particle motion with initial condition is bounded by an ellipse described by $2J_x \approx \varepsilon_x$.

Again, the general solution can be written as

$$x = \sqrt{\beta_x\varepsilon_x}\cos\left(\nu_x\phi_x + \phi_{x0}\right) \tag{88}$$

Using the invariant

$$2J_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \tag{89}$$

Courant-Snyder parameter transform from s_0 to s_1 by the matrix transformation can be represented as

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix} \tag{90}$$

if

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \tag{91}$$

1.14.1 rms Emittance

For a given normalized particle density function, $\rho_x(x,x')$, in phase space,

$$\int \rho_x(x,x')dxdx' = 1. \tag{92}$$

The moments of the beam distribution are

$$\begin{aligned}
<x> &= \int x\rho_x(x,x')dxdx' \\
<x'> &= \int x'\rho_x(x,x')dxdx' \\
\sigma_x^2 &= \int (x-<x>)^2 \rho_x(x,x')dxdx' \\
\sigma_x'^2 &= \int (x'-<x'>)^2 \rho_x(x,x')dxdx' \\
\sigma_{xx'} &= \int (x-<x>)(x'-<x'>)\rho_x(x,x')dxdx'.
\end{aligned} \tag{93}$$

Then, the rms emittance is defined as

$$\varepsilon_x^{\text{rms}} = \sqrt{\sigma_x^2\sigma_x'^2 - \sigma_{xx'}^2}. \tag{94}$$

1.15 Dispersion

So far, we have considered a reference particle moving on the closed orbit with momentum, p_0.

However, a beam is composed of particle distributions with different momenta(off-momentum particles) around the synchronous particle whose momentum is p_0.

First, we define momentum deviation as $\Delta p \equiv p - p_0$, and the momentum spread as $\delta = \Delta p/p_0$ which is the fractional momentum deviation.

Then the equations of motion in lowest order becomes

$$\begin{aligned}
x'' - \left(1+\frac{x}{\rho}\right) &= \frac{B_y}{(B\rho)}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2 \\
&= (B_0 + B_1 x)\frac{1}{(B\rho)}\frac{p_0}{p}\left(1+\frac{1}{2}\frac{x^2}{\rho^2}+\cdots\right) \\
\Rightarrow x'' &+ \left[\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K}{1+\delta}\right]x = \frac{\delta}{\rho(1+\delta)}.
\end{aligned} \tag{95}$$

where

$$K = \frac{B_1}{(B\rho)} = \frac{1}{(B\rho)}\frac{\partial B_y}{\partial x} \tag{96}$$

For $\delta = 0$,

$$x'' + \left(\frac{1}{\rho^2} - K\right)x = 0. \tag{97}$$

The solution of the linearized inhomogeneous equation can be found by setting

$$x = x_\beta(s) + D_x(s)\delta \tag{98}$$

which satisfies

$$\begin{aligned} x_\beta'' + (K_x(s) + \Delta K_x)x_\beta &= 0 \\ D_x'' + (K_x(s) + \Delta K_x)D_x &= \frac{1}{\rho} + \mathcal{O}(\delta) \end{aligned} \tag{99}$$

where

$$\begin{aligned} K_x &= \frac{1}{\rho^2} - K(s) \\ \Delta K_x &= \left[-\frac{2}{\rho^2} + K(s)\right]\delta + \mathcal{O}(\delta^2) \end{aligned} \tag{100}$$

By ignoring chromatic perturbation term, ΔK_x, then

$$D_x'' + K_x(s)D_x = \frac{1}{\rho}. \tag{101}$$

This is the **Dispersion Equation**.

From the fact that $K(s)$ and $\rho(s)$ are periodic, i.e.,

$$D_x(s+L) = D_x(s) \quad \text{and} \quad D_x'(s+L) = D_x'. \tag{102}$$

Again, the solution of the inhomogeneous equation is

$$\begin{aligned} \begin{pmatrix} D_x(s) \\ D_x'(s) \end{pmatrix} &= \mathbf{M}(s|s_0)\begin{pmatrix} D_x(s_0) \\ D_x'(s_0) \end{pmatrix} + \begin{pmatrix} d_x \\ d_x' \end{pmatrix} \\ \Rightarrow \begin{pmatrix} D_x(s) \\ D_x'(s) \\ 1 \end{pmatrix} &= \begin{pmatrix} \mathbf{M}(s|s_0) & \mathbf{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_x(s_0) \\ D_x'(s_0) \\ 1 \end{pmatrix} \end{aligned} \tag{103}$$

For a magnet with constant dipole field and field gradient,

$$\mathbf{d} = \begin{cases} \begin{pmatrix} \frac{1}{\rho K_x}\left(1 - \cos\sqrt{K_x}s\right) \\ \frac{1}{\rho\sqrt{K_x}}\sin\sqrt{K_x}s \end{pmatrix} & \text{if} \quad K_x > 0 \\ \begin{pmatrix} \frac{1}{\rho|K_x|}\left(-1 + \cosh\sqrt{|K_x|}s\right) \\ \frac{1}{\rho\sqrt{|K_x|}}\sinh\sqrt{|K_x}s \end{pmatrix} & \text{if} \quad K_x < 0 \end{cases} \tag{104}$$

Therefore, off-momentum particles follow dispersion trajectories

$$x_D = D_x\frac{\Delta p}{p_0} = D_x\delta \tag{105}$$

which in general are not the same length as the reference orbit.

For a pure dipole, $K_x = 1/\rho^2$, then

$$\begin{aligned} D_x(s) &= \rho(1 - \cos\theta) \approx \rho\theta^2/2 = l\theta/2 \\ D_x(s)' &= \sin\theta \approx \theta \end{aligned} \tag{106}$$

Here, the approximation is again assuming short dipole and $l = \rho\theta$ is the length of the dipole.

1.15.1 Dispersion with Periodic Condition

For a periodic lattice with length L, we must have the dispersion function satisfy

$$D_x(s+L) = D_x(s) \quad \text{and} \quad D'_x(s+L) = D'_x. \tag{107}$$

And the expanded 3 × 3 one turn transfer matrix has

$$\begin{pmatrix} D_x(s_0+C) \\ D'_x(s_0+C) \\ 1 \end{pmatrix} = \begin{pmatrix} D_x(s_0) \\ D'_x(s_0) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s_0+C|s_0) & \mathbf{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_x(s_0) \\ D'_x(s_0) \\ 1 \end{pmatrix} \tag{108}$$

where C is the circumference of the circular accelerator and the matrix $\mathbf{M}(s_0+C|s_0)$ is the one turn betatron transfer matrix:

$$\mathbf{M}(s_0+C|s_0) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \cos\phi_x + \alpha_x \sin\phi_x & \beta_x \sin\phi_x \\ -\gamma_x \sin\phi_x & \cos\phi_x - \alpha_x \sin\phi_x \end{pmatrix} \tag{109}$$

The elements of $\mathbf{d}$ are related to the dispersion function by

$$\mathbf{d} = (\mathbf{I} - \mathbf{M}) \begin{pmatrix} D_x \\ D'_x \end{pmatrix} = \left(I - \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right) \begin{pmatrix} D_x \\ D'_x \end{pmatrix} \tag{110}$$

The matrix $(\mathbf{I} - \mathbf{M})$ has determinant, $2 - M_{11} - M_{22}$. Since the stable condition of the periodic transfer matrix, Trace($\mathbf{M}$) < 2, the determinant of $(\mathbf{I} - \mathbf{M})$ is always positive. Therefore we can calculate the dispersion function from the expanded one turn matrix:

$$\begin{pmatrix} D_x \\ D'_x \end{pmatrix} = (\mathbf{I} - \mathbf{M})^{-1} \mathbf{d} = \frac{1}{2 - M_{11} - M_{22}} \begin{pmatrix} 1 - M_{22} & M_{12} \\ M_{21} & 1 - M_{11} \end{pmatrix} \mathbf{d} \tag{111}$$

1.15.2 $\mathcal{H}$-Function

Since the dispersion function follows the inhomogeneous Hill's function, we can follow the action-angle treatment of the particle motion and define the normalized dispersion as

$$\begin{aligned} X_D &= \frac{D_x}{\sqrt{\beta_x}}, \\ X'_D &= \frac{\alpha_x D_x + \beta_x D'_x}{\sqrt{\beta_x}} \end{aligned} \tag{112}$$

Then dispersion $\mathcal{H}$-function is defined as

$$\begin{aligned} \mathcal{H} &= X_D^2 + X_D'^2 \\ &= \frac{1}{\beta_x} \left[D_x^2 + (\alpha_x D_x + \beta_x D'_x)^2 \right] \\ &= \gamma_x D_x^2 + 2\alpha_x D_x D'_x + \beta_x D_x'^2 \end{aligned} \tag{113}$$

Since the dispersion function satisfies the homogeneous betatron equation of motion in regions without dipoles, i.e., $1/\rho = 0$, the $\mathcal{H}$-function is invariant, but in regions with dipoles, it is not invariant.

Then, the normalized dispersion phase space coordinates can be written as

$$\begin{aligned} X_D &= \frac{D_x}{\sqrt{\beta_x}} = \sqrt{2J_D}\cos\phi_D, \\ X'_D &= \frac{\alpha_x D_x + \beta_x D'_x}{\sqrt{\beta_x}} = -\sqrt{2J_D}\sin\phi_D \end{aligned} \tag{114}$$

where the **dispersion action** is given by

$$J_D = \frac{1}{2}\mathcal{H}(D, D'). \tag{115}$$

In a straight section, J_D is invariant, and ϕ_D is identical to the betatron phase advance.

In a region with dipoles, J_D is not constant. The change of the dispersion function across a thin dipole is $\Delta D = 0$ and $\Delta D' = \theta$, i.e.,

$$\Delta X_D = 0, \Delta X'_D = \sqrt{\beta_x}\Delta D' = \sqrt{\beta_x}\theta, \tag{116}$$

where θ is the bending angle of the dipole, and the change in dispersion action is then

$$\Delta J_D = (\alpha_x D_x + \beta_x D'_x)\,\theta \tag{117}$$

1.16 Achromatic Lattice Design

1.16.1 Achromat

Since the beam closed orbit depends on particle momentum, it would be beneficial to design a transport system that does not depend on beam momentum. Such a beam transport system is called **achromat**.

1.16.2 Achromatic Condition

For a lattice cell represented by a transfer matrix $\mathbf{M}$, the achromatic condition reads that the if the zero dispersion is given at the entrance, zero dispersion is achieved at the exit.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M}(s|s_0) & \mathbf{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{118}$$

Therefore the achromatic condition requires

$$\mathbf{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{119}$$

1.16.3 Dispersion Matching

(**First-Order Achromat Theorem**) A lattice of n-repetitive cells is achromatic to first order iff $M^n = I$ or each cell is achromatic.

If we repeat a non-achromatic cell for n times, the 3×3 matrix becomes

$$\begin{aligned} \mathcal{M}^n &= \begin{pmatrix} \mathbf{M} & \mathbf{d} \\ 0 & 1 \end{pmatrix}^n \\ &= \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^{n-1} + \mathbf{M}^{n-2} + \cdots + 1)\mathbf{d} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{M}^n & (\mathbf{M}^n - \mathbf{I})(\mathbf{M} - \mathbf{I})^{-1}\mathbf{d} \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{120}$$

Therefore, to achieve a achromatic condition by repeating a non-achromatic cell n times, it is required to have

$$\mathbf{M}^n = \mathbf{I} \tag{121}$$

Since the betatron transfer matrix $\mathbf{M}$ can be expressed by the Courant-Snyder parameter

$$\mathbf{M} = \begin{pmatrix} \cos\phi_x + \alpha_x \sin\phi_x & \beta_x \sin\phi_x \\ -\gamma_x \sin\phi_x & \cos\phi_x - \alpha_x \sin\phi_x \end{pmatrix}, \tag{122}$$

the above achromatic condition becomes

$$n\phi_x = 2m\pi \quad \text{for an integer } m \tag{123}$$

1.16.4 Dispersion Suppression

For a non-achromatic lattice in a periodical structure, we have

$$\begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{M} & \mathbf{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} \tag{124}$$

A dispersion suppression section, represented by matrix $\tilde{\mathcal{M}}^{-1}$ can match a non-zero dispersion (D_x, D'_x) at the end of a curved lattice to zero dispersion, which reads

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \tilde{\mathcal{M}}^{-1} \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} D_x \\ D'_x \\ 1 \end{pmatrix} = \tilde{\mathcal{M}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{M}} & \tilde{\mathbf{d}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{d}} \\ 1 \end{pmatrix}. \tag{125}$$

Then, the required terms in the suppressor becomes

$$\tilde{\mathbf{d}} = \begin{pmatrix} D_x \\ D'_x \end{pmatrix} = (\mathbf{I} - \mathbf{M})^{-1}\mathbf{d}. \tag{126}$$

Therefore we can use optimization methods to achieve such condition using any accelerator design software.

1.16.5 Natural Emittance

The **natural emittance** of a storage ring is defined as

$$\varepsilon_0 = C_q \gamma_0^2 \frac{\oint \frac{\mathcal{H}_x}{|\rho^3|} ds}{j_x \oint \frac{ds}{\rho^2}}, \tag{127}$$

where $C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc}$ is the quantum radiation constant and j_x is the horizontal damping partition number.

The natural emittance of an electron synchrotron storage ring often has a significant impact on the performance of the storage ring. In a light source, the natural emittance will affect the brightness; in a collider, it will affect the luminosity.

A FODO cell is a simplest kind of lattice designs that one could consider for a synchrotron storage ring. For a storage ring with FODO cells, the natural emittance is

$$\varepsilon_0 \approx C_q \gamma_0^2 \left(\frac{2f}{L} \right)^3 \theta^3 \tag{128}$$

where f is the focal strength of quadrupoles, L is the dipole length, and θ is the bending angle.

1.16.6 Double Bend Achromat (DBA)

As a first attempt to reduce the natural emittance, one can consider a design in which the dipoles occur in pairs within each cells, with zero dispersion ($D = D' = 0$) at the entrance to the first dipole and at the exit of the second dipole. This system is called a **Double Bend Achromat(DBA)**. The DBA cell can be split into two distinct sections, dispersion free section an dispersion matching section:

$$[\text{Dispersion Free Section}] - \mathbf{B} - [\text{Dispersion Matching Section}] - \mathbf{B} - [\text{Dispersion Free Section}]. \tag{129}$$

Both sections can have a single quad, doublet or triplet. For example, one can construct a DBA cell as follows:

$$\mathbf{O - B - O - QF - O - B - O} \tag{130}$$

The transfer function of the half of the DBA cell is then given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_D & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \tag{131}$$

To have a free dispersion at the entrance of the dipole with the symmetric condition,

$$\begin{pmatrix} D \\ 0 \\ 1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{132}$$

and this gives

$$\begin{aligned} f &= \frac{1}{2}(L_D + \frac{1}{2}L) \\ D &= (L_D + \frac{1}{2}L)\theta. \end{aligned} \tag{133}$$

The natural emittance in a DBA cell is minimized as

$$\varepsilon_0 = \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3, \tag{134}$$

if the lattice functions at the entrance to the first dipole satisfy

$$\begin{aligned} \beta_x &= \sqrt{\frac{12}{5}} L + \mathcal{O}(\theta^3) \\ \alpha_x &= \sqrt{15} + \mathcal{O}(\theta^2). \end{aligned} \tag{135}$$

1.16.7 Multi-Bend Achromat(MBA)

If one construct a MBA cell with M dipoles in which outer dipoles have bending angles of $a\theta$ and inner dipoles have bending angles of $b\theta$, then

$$2a + (M-2)b = M. \tag{136}$$

We need to control the dispersion and the lattice functions to minimize the natural emittance, and when $b = \sqrt[3]{3}a$, the natural emittance is

$$\varepsilon_0 = \frac{C_q \gamma^2}{12\sqrt{15}} \left(\frac{M+1}{M-1} \right) \theta^3. \tag{137}$$

When $M = 2$, this gives

$$\varepsilon_0 = \frac{3 C_q \gamma^2}{12\sqrt{15}} \theta^3 = \frac{C_q \gamma^2}{4\sqrt{15}} \theta^3. \tag{138}$$

1.17 Momentum Compaction Factor

The path length is therefore a function of momentum. In circular accelerators, this leads to a dependence of the revolution period on the particle momentum, which plays an important role in the longitudinal phase focusing of the circulating particles.

The ratio of relative change in path length $\Delta L/L$ to the relative difference in momentum $\Delta p/p_0$ is defined as the **momentum compaction factor**:

$$\alpha_c \equiv \frac{\Delta L / L_0}{\Delta p / p_0} \tag{139}$$

Note that only bending magnets cause any significant change in the path length. An off-momentum particle travels an infinitesimal distance $d\tilde{s}$ through a bending magnet,

$$d\tilde{s} = \frac{\rho + x_D}{\rho} ds \tag{140}$$

if the on-momentum particle's path length is ds. Then the total path length of the off-momentum particle can be written as

$$L = L_0 + \Delta L = \oint d\tilde{s} = \oint \frac{\rho + x_D}{\rho} ds = \oint ds + \frac{\Delta p}{p} \oint \frac{D(s)}{\rho(s)} ds \tag{141}$$

where $L_0 = \oint ds$.

Therefore, the path length change due to the dispersive trajectory is given by

$$\Delta L = \frac{\Delta p}{p} \oint \frac{D(s)}{\rho(s)} ds. \tag{142}$$

And the momentum compaction factor can also be calculated as

$$\alpha_c = \frac{1}{L_0} \oint \frac{D(s)}{\rho(s)} ds. \tag{143}$$

1.18 Transition Energy and the Phase Slip Factor

Particles with different momenta travel along different paths in a circular accelerator. Since the revolution period is $T = C/v$ where C is the circumference and v is the speed of the circulating particles,

$$\frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = \left(\alpha_c - \frac{1}{\gamma^2}\right) \delta \tag{144}$$

Then, we define the **phase slip factor**, η as

$$\eta \equiv \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \tag{145}$$

Here, $\gamma_T \equiv \sqrt{1/\alpha_c}$ is called the **transition-**γ and $\gamma_T mc^2$, or simply γ_T is the **transition energy**.

The momentum compaction factor is determined by the design of the beam line, whereas the Lorentz relativistic factor, γ, depends on the beam energy. Therefore, the phase slip factor could be positive, zero, or negative, depending on the design of the beam line and the beam energy.

1) **Above transition** ($\gamma > \gamma_T$): In a storage ring, the momentum compaction factor is usually positive. Since the circumference of a particle trajectory generally increases with the momentum of the particle. Therefore, the phase slip factor in a storage ring is usually positive for beams of sufficiently high energy. Such a ring is said to be operating 'above transition'. Above transition, the revolution period increases with momentum, because the the increase in speed of a particle is insufficient to compensate the increase in path length.

2) **Below transition** ($\gamma > \gamma_T$): At low energy, the phase slip factor is negative. A storage ring in this regime is said to be operating 'below transition', in which case the revolution period decreases with increasing momentum. The speed of a particle in a storage ring below transition is sufficiently low compared to the speed of light, that increasing the momentum leads to an increase in speed that more than compensates the increase in path length around the ring.

3) **Isochronous condition** ($\gamma = \gamma_T$): At ($\gamma = \gamma_T$), the revolution period is independent of the particle momentum. All particles at different momenta travel rigidly around the accelerator with equal revolution frequencies.

Electron storage rings tend to operate purely above transition, event at relatively low energy. This is in contrast to proton storage rings: because the proton mass is much larger than the that of the electron, the relativistic factor for electrons at a given energy will be much larger than that for protons at the same energy.

1.19 Perturbed Motion Due to Linear Magnet Errors

In the presence of magnetic field errors,

$$\begin{aligned} x'' + K_x(s)x = 0 \quad &\Rightarrow \quad x'' + K_x(s)x = \tfrac{\Delta B_y}{B\rho} \\ y'' + K_y(s)y = 0 \quad &\Rightarrow \quad y'' + K_y(s)y = -\tfrac{\Delta B_x}{B\rho} \end{aligned} \tag{146}$$

where the perturbed magnetic fields can be represented as

$$\Delta B_y + i\Delta B_x = B_0 \sum_{n=0}^{\infty} (b_n + ia_n)(x+iy)^n \tag{147}$$

Here B_0 is the dipole field strength, and b_0, b_1 are dipole and quadrupole field errors, respectively, and b_2 are sextupole filed error, and a_n are skew magnetic field errors.

1.19.1 Dipole Field Errors

For simplicity, we consider a field error that has only a vertical field component, and we assume that there is no coupling between the horizontal and either the vertical or longitudinal motions, so that the vertical motion can be considered independently of motion in the other degrees of freedom.

Staring with the trajectory of a single particle in terms of action-angle variables:

$$\begin{aligned} x &= \sqrt{2\beta_x J_x}\cos\phi_x \\ x' &= -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\phi_x + \alpha_x\cos\phi_x\right) \end{aligned} \tag{148}$$

A vertical magnetic dipole field at some point in the ring will cause a change in x' as the particle passes that point. Then, the closed orbit is no longer the reference trajectory. Suppose that the dipole field is located at a single point $s = s_0$ in the ring, and the it causes a change $\Delta x'$ in x' as a particle passes that point,

$$\Delta x' \approx \frac{q}{p_0}\int B_y ds \tag{149}$$

where p_0 is the reference momentum. The conditions for the particle trajectory to close on itself can be written as

$$\begin{aligned} x(s_0 + C) &= x(s_0) \\ x'(s_0 + C) + \Delta x' &= x'(s_0) \end{aligned} \tag{150}$$

where s_0 is the point immediately after the dipole field error. Let J_{x0} and ϕ_{x0} be the action and angle of a particle on the close orbit immediately after the vertical dipole field. If μ_x is the total phase advance over one turn of the ring, then the close orbit condition can be written in terms of action-angle variables

$$\begin{aligned} \sqrt{2\beta_x J_{x0}}\cos\left(\phi_{x0} + \mu_x\right) &= \sqrt{2\beta_x J_{x0}}\cos\phi_{x0} \\ -\sqrt{\frac{2J_{x0}}{\beta_x}}\left[\sin\left(\phi_{x0} + \mu_x\right) + \alpha_x\cos\left(\phi_{x0} + \mu_x\right)\right] + \Delta x' &= -\sqrt{\frac{2J_{x0}}{\beta_x}}\left(\sin\phi_{x0} + \alpha_x\cos\phi_{x0}\right) \end{aligned} \tag{151}$$

where the Courant-Snyder parameters are to be calculated at $s = s_0$. These implies that

$$\begin{aligned} J_{x0} &= \frac{\beta_x \Delta x'^2}{8 \sin^2(\pi\nu_x)} \\ \phi_{x0} &= -\pi\nu_x, \end{aligned} \tag{152}$$

where $\nu_x = \mu_x/2\pi$ is the horizontal betatron tune.

Therefore, one can see that if the tune ν_x is an integer, then the action is infinite, i.e., no close orbit exist. If the tune is close to an integer, then $\sin(\pi\nu_x)$ will be small, and even a small deflection $\Delta x'$ can lead to a large distortion of the closed orbit. Note that the size of the closed orbit distortion depends on the betatron function at the location of the field error. If the betatron function is large, then a small deflection can again lead to a large closed orbit distortion.

One can write an expression for the closed orbit at a location s, resulting from a single dipole field error at $s = s_0$,

$$x_{\mathrm{co}}(s) = \frac{\sqrt{\beta_x(s)\beta_x(s_0)}}{2\sin(\pi\nu_x)} \Delta x' \cos[\mu_x(s, s_0) - \pi\nu_x], \tag{153}$$

where $\mu_x(s, s_0)$ is the horizontal phase advance from s_0 to s.

If there are a large number of errors distributed around the ring, the close orbit is given by

$$x_{\mathrm{co}}(s) = \oint_C \frac{\sqrt{\beta_x(s)\beta_x(s')}}{2\sin(\pi\nu_x)} \frac{q}{p_0} B_y(s') \cos[\mu_x(s, s') - \pi\nu_x] ds', \tag{154}$$

where C is the length of the reference trajectory, i.e., the circumference of the ring, and $B_y(s)$ is a vertical dipole field error at the location s.

1.19.2 Quadrupole Field Errors

The gradient error of the quadrupole contributes to the perturbation of the betatron tune. Starting from the Hill's equation with the delta function response,

$$x''(s) + K_{x0}(s)x(s) = -(k_{x1}ds_1)\delta(s - s_0)x(s) \tag{155}$$

$$\Rightarrow \quad x''(s) + [K_{x0}(s) + k_{x1}ds_1\delta(s - s_0)]\, x(s) = 0. \tag{156}$$

We can easily find the tune change from the perturbed transfer matrix as

$$\begin{aligned} \mathbf{M}_{\text{perturbed}} &= \mathbf{M}(s_0)\mathbf{m}(s_1) \\ &= \begin{pmatrix} \cos\phi_{x0} + \alpha_{x0}\sin\phi_{x0} & \beta_{x0}\sin\phi_{x0} \\ -\gamma_{x0}\sin\phi_{x0} & \cos\phi_{x0} - \alpha_{x0}\sin\phi_{x0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k_{x1}ds_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi_{x0} + \alpha_{x0}\sin\phi_{x0} - \beta_{x0}k_{x1}ds_1\sin\phi_{x0} & \beta_{x0}\sin\phi_{x0} \\ -\gamma_{x0}\sin\phi_{x0} - k_{x1}ds_1(\cos\phi_{x0} + \alpha_{x0}\sin\phi_{x0}) & \cos\phi_{x0} - \alpha_{x0}\sin\phi_{x0} \end{pmatrix} \end{aligned} \tag{157}$$

The phase advance of the perturbed motion can be obtained from the trace of $\mathbf{M}$,

$$2\cos\phi_x = 2\cos\phi_{x0} - \beta_{x0}k_{x1}ds_1\sin\phi_{x0} \tag{158}$$

If the tune change is small, then the phase advance change is given by

$$\Delta\phi_x = \frac{1}{2}\beta_{x0}k_{x1}ds_1. \tag{159}$$

Thus the betatron tune shift is

$$\Delta\nu_x = \frac{1}{4\pi}\beta_{x0}k_{x1}ds_1 \tag{160}$$

Therefore, the betatron tune shift due to the distributed field gradient error is

$$\Delta\nu_x = \frac{1}{4\pi}\oint_C \beta_x(s)k_x(s)ds \tag{161}$$

1.20 Chromaticity and Compensation

1.20.1 Chromatic Aberrations and Chromaticity

Chromatic effects are caused by the momentum dependence of the focal properties of lattice elements. From the momentum deviation, $\delta = \frac{\Delta p}{p_0} = \frac{p-p_0}{p_0}$, a higher energy particle with $\delta > 0$ has a larger momentum rigidity and thus a weaker effective focusing strength.

Consider two particles with momentum p_- and p_+, where $p_- < p_0 < p_+$, i.e., $\frac{p_- - p_0}{p_0} = \delta_- < 0 < \delta_+ = \frac{p_+ - p_0}{p_0}$. The bending radii of these particles can be calculated from the definition of the magnetic rigidity, $B\rho = p/q$,

$$\frac{1}{B}\frac{p_-}{q} = \rho_- < \rho_0 = \frac{1}{B}\frac{p_0}{q} < \rho_+ = \frac{1}{B}\frac{p_+}{q}. \tag{162}$$

Therefore, one can note that there is a systematic gradient error, $\Delta K_{x,y}$, which are given by

$$\begin{aligned} \Delta K_x &= \left[-\frac{2}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\right]\delta + \mathcal{O}(\delta^2) \approx -K_x\delta \\ \Delta K_x &= -\frac{1}{B\rho}\frac{\partial B_y}{\partial x}\delta + \mathcal{O}(\delta^2) \approx -K_y\delta \end{aligned} \tag{163}$$

The dependence of the focusing strength on the momentum of a circulating particle is called **chromatic aberration**. Therefore, the gradient error induces betatron tune shift and betatron amplitude function perturbation.

$$\Delta\nu_x = \frac{1}{4\pi}\oint \beta_x \Delta K_x ds \approx -\left(\frac{1}{4\pi}\oint \beta_x K_x ds\right)\delta. \tag{164}$$

Then, the **chromaticity**, defined as the derivatives of the betatron tune over the fractional momentum deviation, is

$$C_x(=\xi_x) \equiv \frac{d(\Delta\nu_x)}{d\delta}. \tag{165}$$

For the contributions only with quadrupoles in a lattice, it is called the **natural chromaticity**.

$$\begin{aligned} C_{x,\mathrm{nat}} &= -\frac{1}{4\pi}\oint \beta_x K_x ds, \\ C_{y,\mathrm{nat}} &= -\frac{1}{4\pi}\oint \beta_y K_y ds. \end{aligned} \tag{166}$$

For a focusing quadrupole, $K_{x,y} > 0$, i.e., $\oint \beta_{x,y}K_{x,y}ds > 0$, $C_{x,y} > 0$. In other words, the focusing function is weaker for high energy particles, the betatron tune decreases with particle momentum, and the natural chromaticity is negative.

1.21 Chromaticity Correction

A large negative natural chromaticity leads a large tune spread, and consequently a short storage life time. Therefore, chromatic correction is needed for the storage ring design. We need a magnet whose focusing strength increases linearly with momentum.

For a sextupole,

$$\begin{aligned} \frac{\Delta B_y}{B\rho} &= \frac{1}{2!}\frac{B_2}{B\rho}(x^2-y^2) = \frac{1}{2}\frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}(x^2-y^2), \\ \frac{\Delta B_x}{B\rho} &= \frac{1}{2!}\frac{B_2}{B\rho}xy = \frac{1}{2}\frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}xy, \end{aligned} \tag{167}$$

where $\frac{\partial^2 B_y}{\partial x^2}$ is evaluated on the closed orbit ($x = 0$ and $y = 0$).

By substituting $x = x_\beta + D_x\delta$ and $y = y_\beta$,

$$\begin{aligned} \frac{\Delta B_y}{B\rho} &= \frac{1}{2}\frac{B_2}{B\rho}\left[(x_\beta + D_x\delta)^2 - y_\beta^2\right] \\ &= \frac{B_2}{B\rho}x_\beta D_x\delta + \frac{1}{2}\frac{B_2}{B\rho}\left(x_\beta^2 - y_\beta^2\right) + \frac{1}{2}\frac{B_2}{B\rho}D_x^2\delta^2 \\ &= -(S_x D_x\delta)x_\beta - \frac{S_x}{2}\left(x_\beta^2 - y_\beta^2\right) - \frac{S_x}{2}D_x^2\delta^2, \end{aligned} \tag{168}$$

where $S_x = -\frac{B_2}{B\rho} = -\frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}$, is the **effective sextupole strength**.

Similarly,

$$\frac{\Delta B_x}{B\rho} = -(S_x D_x\delta)y_\beta - S_x x_\beta y_\beta. \tag{169}$$

Then, we can define the effective quadrupole gradient as

$$\begin{aligned} \Delta K_x &= S_x D_x \delta, \\ \Delta K_y &= -S_x D_x \delta. \end{aligned} \tag{170}$$

which depends linearly on the momentum deviation, δ, therefore, sextupole can be used for chromaticity correction.

The chromaticity including the contributions from sextupoles is then given by

$$\begin{aligned} C_x &= -\frac{1}{4\pi}\oint \beta_x\left(K_x - S_x D_x\right)ds, \\ C_y &= -\frac{1}{4\pi}\oint \beta_y\left(K_y + S_x D_x\right)ds. \end{aligned} \tag{171}$$

This shows that sextupoles located at non-zero dispersion region can correct chromaticity, and those are called chromatic sextupoles.

1.22 Summary

In the presence of magnetic fields, the generalized Hill's equation can be written as

$$x'' - \frac{\rho + x}{\rho^2} = \pm\frac{B_y}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 \tag{172}$$

To the first order,

$$x'' + \left[\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K_{1x}(s)}{1+\delta}\right] x = \frac{\delta}{\rho(1+\delta)} \tag{173}$$

where $K_{1x}(s) = \frac{B_1}{B\rho} = \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\Big|_{x=0,y=0}$.

1) **Betatron Motions**: on-momentum particle

For $\delta = 0$,

$$x'' + \left(\frac{1}{\rho} - K_{1x}\right) x = x'' + K_x(s)x = 0 \tag{174}$$

where $K_x = \frac{1}{\rho^2} - \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\Big|_{x=0,y=0}$.

2) **Dispersion Motions**: off-momentum particle

For $\delta \neq 0$, $x = x_\beta(s) + x_D(s) = x_\beta(s) + D_x(s)\delta$ where $\delta = (p-p_0)/p_0 = \Delta p/p_0$.

$$\begin{aligned} x''_\beta + (K_x + \Delta K_x)\, x_\beta &= 0 \\ D''_x + (K_x + \Delta K_x)\, D_x &= \frac{1}{\rho} + \mathcal{O}(\delta) \end{aligned} \tag{175}$$

where $K_x = \frac{1}{\rho^2} - \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\Big|_{x=0,y=0}$ and $\Delta K_x = \left[-\frac{2}{\rho^2} + K_{1x}(s)\right]\delta + \mathcal{O}\left(\delta^2\right)$.

3) **Perturbed Motions**: dipole and quadrupole field errors

For the perturbed motions, the equation of motion becomes

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho} \tag{176}$$

where $\Delta B_y + i\Delta B_x = B_0 \sum_{n=0}^{\infty}(b_n + ia_n)(x+iy)^n$.

Dipole field errors result in closed orbit distortions, which is given by

$$x_{\text{co}}(s) = \oint_C \frac{\sqrt{\beta_x(s)\beta_x(s')}}{2\sin(\pi\nu_x)}\frac{q}{p_0} B_y(s')\cos\left[\mu_x(s,s') - \pi\nu_x\right]ds', \tag{177}$$

where C is the length of the reference trajectory, i.e., the circumference of the ring, and $B_y(s)$ is a vertical dipole field error at the location s.

The betatron tune shift due to the distributed field gradient error is

$$\Delta\nu_x = \frac{1}{4\pi}\oint_C \beta_x(s)k_x(s)ds \tag{178}$$

4) **Chromatic Aberrations**: systematic gradient errors

Systematic gradient errors of linear magnet elements, $\Delta K_{x,y}$, are given by

$$\begin{aligned} \Delta K_x &= \left[-\frac{2}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x}\right]\delta + \mathcal{O}(\delta^2) \approx -K_x\delta \\ \Delta K_x &= -\frac{1}{B\rho}\frac{\partial B_y}{\partial x}\delta + \mathcal{O}(\delta^2) \approx -K_y\delta \end{aligned} \tag{179}$$

The **chromaticity**, is defined as the derivatives of the betatron tune over the fractional momentum deviation:

$$C_x (= \xi_x) \equiv \frac{d(\Delta \nu_x)}{d\delta}. \tag{180}$$

For the contributions only with quadrupoles in a lattice, it is called the **natural chromaticity**.

$$\begin{aligned} C_{x,\mathrm{nat}} &= -\frac{1}{4\pi} \oint \beta_x K_x ds, \\ C_{y,\mathrm{nat}} &= -\frac{1}{4\pi} \oint \beta_y K_y ds. \end{aligned} \tag{181}$$

References

- H. Weidemann, Particle Accelerator Physics, 4th Ed, SPringer (2015)
- S. Y. Lee, “Accelerator Physics,” 1st Edition, World Scientific (1999)