3. Foundation of Beam Dynamics

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1 Introduction

1.1 Learning Objectives

By the end of this lecture, students should be able to:

- explain why Hamiltonian mechanics is the natural framework for accelerator beam dynamics,
- distinguish between **mechanical momentum** and **canonical momentum**,
- derive the relativistic Lagrangian and Hamiltonian of a charged particle in electromagnetic fields,
- write Hamilton's equations for charged-particle motion,
- understand the need for a **reference particle** and a **reference trajectory**,
- describe the role of **curvilinear coordinates** in accelerator physics,
- understand the meaning of **phase space** and **symplecticity**,
- derive the leading Hamiltonian forms of magnetic multipoles from canonical vector potentials,
- recognize how the Hamiltonian formalism leads naturally to linear and nonlinear beam optics.

1.2 Motivation: From Fields to Particle Motion

In the previous lectures, we studied:

1. what accelerators are and how they are used,
2. the mechanics, electromagnetism, and special relativity needed for charged-particle motion,
3. the electromagnetic fields of accelerator devices such as dipoles, quadrupoles, sextupoles, waveguides, and rf cavities.

The next step is the central dynamical question:

> Once the electromagnetic fields are known, how does a charged particle move through the accelerator?

This is the beginning of **beam dynamics**.

At the most basic level, beam dynamics concerns:

- how a single particle moves through accelerator elements,
- how that motion is described relative to a design orbit,
- how focusing and acceleration are achieved,
- how stable motion is maintained.

In a broader sense, beam dynamics also includes:

- the evolution of a beam distribution in phase space,
- coupling between transverse and longitudinal motion,
- collective effects such as space charge, wakefields, and instabilities.

This lecture begins with the **single-particle Hamiltonian formulation**, because it is the foundation on which most of accelerator beam dynamics is built. The Symon note develops the complete accelerator Hamiltonian in curvilinear coordinates and then derives canonical potentials for lattice elements, which is exactly the viewpoint adopted here.

1.3 Why Hamiltonian Mechanics?

A relativistic charged particle obeys the Lorentz-force equation

$$\frac{d\Pi}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

where the **mechanical momentum** is

$$\Pi = \gamma m \mathbf{v}.$$

This equation is exact, but in accelerator physics it is not the most convenient starting point for optics and lattice theory.

Hamiltonian mechanics is preferred because:

1. it is formulated in terms of **canonical coordinates** and **canonical momenta**,
2. it naturally describes motion in **phase space**,
3. it preserves **symplectic structure**,
4. it is well suited for transfer matrices and maps,
5. it generalizes naturally to nonlinear dynamics and perturbation theory,
6. it provides a clear bridge to numerical tracking methods.

In accelerator physics, the motion through drifts, dipoles, quadrupoles, rf cavities, and nonlinear elements is most naturally described as a Hamiltonian system.

1.4 Review of Relativistic Charged-Particle Dynamics

1.4.1 Relativistic kinematics

The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad \beta = \frac{v}{c}.$$

The relativistic mechanical momentum is

$$\Pi = \gamma m \mathbf{v}.$$

The total energy is

$$E = \gamma mc^2.$$

The kinetic energy is

$$K = (\gamma - 1)mc^2.$$

These satisfy the energy-momentum relation

$$E^2 = \Pi^2 c^2 + m^2 c^4.$$

1.4.2 Lorentz force and potentials

The electric and magnetic fields are expressed in terms of the scalar and vector potentials as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The Lorentz-force equation is

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

This is the direct dynamical equation, but we now seek a Lagrangian and Hamiltonian formulation.

1.5 Derivation of the Relativistic Lagrangian for a Charged Particle in Electromagnetic Fields

1.5.1 Start from the Lagrangian Equation

For a Lagrangian $L(\mathbf{r}, \mathbf{v}, t)$, the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

We want this equation to reproduce the relativistic Lorentz-force law

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Thus, our task is to construct a Lagrangian whose Lagrangian equation gives exactly this force law.

1.5.2 Derivation of the Free Relativistic Lagrangian

We first consider a free particle, with no electromagnetic field.

For a free particle, the momentum must be

$$\mathbf{p} = \gamma m \mathbf{v}.$$

In Lagrangian mechanics, the canonical momentum is

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}.$$

So for a free relativistic particle, we require

$$\frac{\partial L_{\text{free}}}{\partial \mathbf{v}} = \gamma m \mathbf{v}.$$

Since the free-particle Lagrangian depends only on v^2, let

$$L_{\text{free}} = L_{\text{free}}(v^2).$$

Then

$$\frac{\partial L_{\text{free}}}{\partial \mathbf{v}} = \frac{dL_{\text{free}}}{d(v^2)} \cdot 2\mathbf{v}.$$

We want this to equal

$$\gamma m \mathbf{v}.$$

So we must have

$$2\frac{dL_{\text{free}}}{d(v^2)} = \gamma m,$$

or

$$\frac{dL_{\text{free}}}{d(v^2)} = \frac{m}{2\sqrt{1 - v^2/c^2}}.$$

Now integrate with respect to v^2. Let

$$u = 1 - \frac{v^2}{c^2}.$$

Then

$$d(v^2) = -c^2\, du.$$

So

$$L_{\text{free}} = \int \frac{m}{2\sqrt{1 - v^2/c^2}}\, d(v^2) = \int \frac{m}{2\sqrt{u}}(-c^2)\, du.$$

Thus,

$$L_{\text{free}} = -\frac{mc^2}{2} \int u^{-1/2}\, du = -\frac{mc^2}{2}(2u^{1/2}) + \text{constant}.$$

Therefore,

$$L_{\text{free}} = -mc^2\sqrt{u} + \text{constant}.$$

Substituting back $u = 1 - v^2/c^2$,

$$L_{\text{free}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \text{constant}.$$

We usually drop the additive constant, since it does not affect the equations of motion. Hence,

$$L_{\text{free}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}.$$

1.5.3 Check the Free-Particle Momentum

Differentiate the free-particle Lagrangian:

$$L_{\text{free}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}.$$

Then

$$\frac{\partial L_{\text{free}}}{\partial \mathbf{v}} = -mc^2 \cdot \frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot \left(-\frac{2\mathbf{v}}{c^2}\right).$$

This becomes

$$\frac{\partial L_{\text{free}}}{\partial \mathbf{v}} = m\frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\mathbf{v}.$$

So the free-particle Lagrangian is correct.

1.5.4 Add Electromagnetic Interaction

Now we want to include electromagnetic fields.

The electric and magnetic fields are expressed in terms of the scalar and vector potentials as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

We seek an interaction Lagrangian L_{int} such that the full Lagrangian equations reproduce the Lorentz force.

The correct interaction term is

$$L_{\text{int}} = q\mathbf{A} \cdot \mathbf{v} - q\phi.$$

So the full Lagrangian is

$$L = L_{\text{free}} + L_{\text{int}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{A} \cdot \mathbf{v} - q\phi.$$

Now we verify that this gives the Lorentz force.

1.5.5 Compute the Canonical Momentum

From

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{A} \cdot \mathbf{v} - q\phi,$$

the canonical momentum is

$$\frac{\partial L}{\partial \mathbf{v}} = \gamma m\mathbf{v} + q\mathbf{A}.$$

So

$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A}.$$

Now apply the Lagrangian equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

This gives

$$\frac{d}{dt}(\gamma m\mathbf{v} + q\mathbf{A}) - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

Expand the time derivative:

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{d\mathbf{A}}{dt} - \frac{\partial L}{\partial \mathbf{r}} = 0.$$

1.5.6 Compute $\partial L/\partial \mathbf{r}$

The free-particle term does not depend explicitly on $\mathbf{r}$, so only the interaction terms contribute:

$$L_{\text{int}} = q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

Therefore,

$$\frac{\partial L}{\partial \mathbf{r}} = q\nabla(\mathbf{A}\cdot\mathbf{v}) - q\nabla\phi.$$

Since $\mathbf{v}$ is independent of $\mathbf{r}$ in the partial derivative,

$$\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}).$$

Also,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}.$$

Substitute these into the Lagrangian equation:

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}\right) - q\left[(\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A})\right] + q\nabla\phi = 0.$$

The $(\mathbf{v}\cdot\nabla)\mathbf{A}$ terms cancel, leaving

$$\frac{d}{dt}(\gamma m\mathbf{v}) + q\frac{\partial \mathbf{A}}{\partial t} - q\mathbf{v}\times(\nabla\times\mathbf{A}) + q\nabla\phi = 0.$$

Rearrange:

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A})\right).$$

Now use

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}.$$

Then

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

This is exactly the Lorentz-force law.

1.5.7 Final Result

Thus the Lagrangian

$$L(\mathbf{r}, \mathbf{v}, t) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{A} \cdot \mathbf{v} - q\phi$$

is the correct relativistic Lagrangian for a charged particle in electromagnetic fields because:

1. the free-particle part gives the correct relativistic momentum,
2. the interaction terms produce the Lorentz force through the Lagrangian equations.

1.5.8 Physical Interpretation of Each Term

The Lagrangian has three parts:

Free relativistic particle

$$-mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

This term describes the relativistic inertial motion of the particle.

Coupling to the vector potential

$$q\mathbf{A} \cdot \mathbf{v}$$

This term is responsible for the magnetic interaction and part of the electric interaction.

Coupling to the scalar potential

$$-q\phi$$

This term is the electrostatic potential-energy contribution.

Together, these terms reproduce the full electromagnetic force.

1.5.9 Nonrelativistic Limit

As a useful check, consider the limit $v \ll c$. Then

$$\sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}.$$

So

$$-mc^2\sqrt{1-\frac{v^2}{c^2}} \approx -mc^2 + \frac{1}{2}mv^2.$$

Thus the Lagrangian becomes

$$L \approx -mc^2 + \frac{1}{2}mv^2 + q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

Dropping the constant $-mc^2$, which does not affect the equations of motion, we recover the familiar nonrelativistic electromagnetic Lagrangian:

$$L_{\text{nr}} = \frac{1}{2}mv^2 + q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

This is a good consistency check.

1.6 Lagrangian Formulation in Electromagnetic Fields

The relativistic Lagrangian for a particle of mass m and charge q in electromagnetic fields is

$$L(\mathbf{r},\mathbf{v},t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} + q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

This Lagrangian contains three parts:

1. the free-particle term
$$-mc^2\sqrt{1-\frac{v^2}{c^2}},$$
2. the coupling to the vector potential
$$q\mathbf{A}\cdot\mathbf{v},$$
3. the coupling to the scalar potential
$$-q\phi.$$

The potentials rather than the fields appear directly in the Lagrangian and Hamiltonian formalisms. This is why canonical momentum differs from mechanical momentum in electromagnetic systems.

2 Hamiltonians in Accelerators

2.1 Canonical Momentum and Its Physical Meaning

The canonical momentum is defined by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}.$$

Carrying out the derivative gives

$$\mathbf{p} = \gamma m \mathbf{v} + q\mathbf{A}.$$

Thus:

2.1.1 Mechanical momentum

$$\Pi = \gamma m \mathbf{v}$$

2.1.2 Canonical momentum

$$\mathbf{p} = \gamma m \mathbf{v} + q\mathbf{A}$$

Hence,

$$\mathbf{p} = \Pi + q\mathbf{A}.$$

This distinction is central in beam dynamics:

- Π is associated with actual motion,
- $\mathbf{p}$ is the momentum entering Hamilton's equations and symplectic maps.

2.2 Hamiltonian Derivation

The Hamiltonian is defined by the Legendre transform

$$H(\mathbf{r}, \mathbf{p}, t) = \mathbf{v} \cdot \mathbf{p} - L.$$

Substituting the relativistic Lagrangian and canonical momentum yields

$$H = \gamma m c^2 + q\phi.$$

This is the Hamiltonian in terms of velocity.

To express it in canonical variables, use

$$\Pi = \gamma m \mathbf{v} = \mathbf{p} - q\mathbf{A}.$$

Then the energy-momentum relation gives

$$(\gamma m c^2)^2 = m^2 c^4 + c^2 (\mathbf{p} - q\mathbf{A})^2.$$

Hence the Hamiltonian becomes

$$H(\mathbf{r}, \mathbf{p}, t) = \sqrt{m^2 c^4 + c^2 (\mathbf{p} - q\mathbf{A})^2} + q\phi.$$

This is the standard Hamiltonian for a relativistic charged particle in electromagnetic fields.

2.3 Hamilton's Equations and Relation to the Lorentz Force

Given $H(q_i, p_i, t)$, Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

In vector form,

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}}.$$

These equations are fully equivalent to the Lorentz-force law, but written in canonical variables.

If the Hamiltonian has no explicit time dependence, it is conserved. In accelerator systems, rf cavities and ramped magnets may introduce explicit time dependence, but the Hamiltonian formalism remains valid.

3 Curvilinear Coordinate Systems

3.1 Choice of Independent Variable: t vs s

In ordinary mechanics, time t is the independent variable. In accelerator physics, it is often more convenient to use the distance along the design orbit,

$$s,$$

as the independent variable.

This is natural because accelerator elements are arranged along a beamline or ring, and the beam is most naturally described as it advances through the machine.

This choice is especially convenient for:

- transfer matrices,
- lattice functions,
- beam transport,
- storage-ring optics,
- map-based tracking.

3.2 Reference Particle and Reference Trajectory

A real beam contains many particles with slightly different coordinates and momenta. We therefore define a hypothetical **reference particle** that:

- has design momentum p_0,
- follows the design orbit,
- arrives at the intended rf phase,
- defines the nominal machine motion.

The path followed by this reference particle is the **reference trajectory**.

Actual particle coordinates are then written as deviations from this trajectory.

Without a reference trajectory, quantities such as transverse displacement, momentum deviation, dispersion, and phase error are not naturally defined.

3.3 Curvilinear Coordinates for Accelerator Motion

Because the reference path is generally curved, Cartesian coordinates are not the most natural choice.

Instead, we use coordinates attached to the reference orbit:

- s: distance along the reference path,
- x: horizontal displacement from the reference orbit,
- y: vertical displacement from the reference orbit.

This moving coordinate system is often identified with the **Frenet-Serret** frame.

It allows the design curvature to be built into the geometry and makes the derivation of beam-optical equations more transparent.

3.4 Paraxial Approximation and Small Deviations

In most accelerators, beam motion remains close to the reference orbit, so we assume

$$|x|, |y| \ll \rho, \qquad |x'|, |y'| \ll 1,$$

where ρ is a characteristic bending radius.

This is the paraxial approximation.

Under this approximation:

- the Hamiltonian can be expanded in powers of x, y, p_x, and p_y,
- only low-order terms need be kept initially,
- linear optics emerges naturally.

This is the starting point for Hill's equation and transfer-matrix optics.

3.5 Phase Space in Accelerator Physics

A common six-dimensional phase-space choice is

$$(x, p_x, y, p_y, z, \delta),$$

where

- x, y: transverse displacements,
- p_x, p_y: canonical transverse momenta,
- z: longitudinal displacement relative to the reference particle,
- δ: relative momentum deviation.

A standard definition is

$$\delta = \frac{\Delta p}{p_0}.$$

Different texts use slightly different longitudinal coordinates, but the physical meaning is always relative motion around a reference particle in six-dimensional phase space.

3.6 Symplecticity and Its Importance

A key structural property of Hamiltonian motion is symplecticity.

For a linear map R, the symplectic condition is

$$R^T S R = S, \qquad S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Symplecticity means:

- canonical phase-space structure is preserved,
- Hamiltonian motion remains consistent under the map,
- artificial damping or growth is not introduced,
- long-term tracking remains physically meaningful.

This is essential in accelerator physics, where particles may circulate for enormous numbers of turns.

3.7 Differential Operators in s-Dependent Accelerator Coordinates

3.7.1 Geometry and Metric Factor

Let the reference orbit have curvature

$$\frac{1}{\rho(s)}.$$

At horizontal displacement x, the longitudinal scale factor becomes

$$h(x,s) = 1 + \frac{x}{\rho(s)}.$$

Thus the line element is

$$d\ell^2 = dx^2 + dy^2 + h^2 ds^2.$$

So the orthogonal scale factors are

$$h_x = 1, \qquad h_y = 1, \qquad h_s = h = 1 + \frac{x}{\rho(s)}.$$

The associated unit vectors are

$$\hat{\mathbf{x}}, \qquad \hat{\mathbf{y}}, \qquad \hat{\mathbf{s}}.$$

This is the standard orthogonal curvilinear system used in accelerator beam dynamics.

3.7.2 Gradient Operator

For any scalar function $f(x, y, s)$, the gradient in orthogonal curvilinear coordinates is

$$\nabla f = \hat{\mathbf{x}} \frac{1}{h_x} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{1}{h_y} \frac{\partial f}{\partial y} + \hat{\mathbf{s}} \frac{1}{h_s} \frac{\partial f}{\partial s}.$$

Since

$$h_x = h_y = 1, \qquad h_s = h,$$

we obtain

$$\nabla f = \hat{\mathbf{x}} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{\partial f}{\partial y} + \hat{\mathbf{s}} \frac{1}{h} \frac{\partial f}{\partial s}.$$

Therefore, the gradient operator itself is

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{s}} \frac{1}{h} \frac{\partial}{\partial s}$$

or explicitly,

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{s}} \frac{1}{1 + x/\rho} \frac{\partial}{\partial s}.$$

3.7.3 Divergence Operator

Let

$$\mathbf{A} = A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_s \hat{\mathbf{s}}.$$

In orthogonal curvilinear coordinates, the divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{h_x h_y h_s} \left[\frac{\partial}{\partial x}(h_y h_s A_x) + \frac{\partial}{\partial y}(h_s h_x A_y) + \frac{\partial}{\partial s}(h_x h_y A_s) \right].$$

Using

$$h_x = h_y = 1, \qquad h_s = h,$$

this becomes

$$\nabla \cdot \mathbf{A} = \frac{1}{h}\left[\frac{\partial}{\partial x}(hA_x) + \frac{\partial}{\partial y}(hA_y) + \frac{\partial A_s}{\partial s}\right].$$

So,

$$\nabla \cdot \mathbf{A} = \frac{1}{h}\left[\frac{\partial}{\partial x}(hA_x) + \frac{\partial}{\partial y}(hA_y) + \frac{\partial A_s}{\partial s}\right]$$

or explicitly,

$$\nabla \cdot \mathbf{A} = \frac{1}{1+x/\rho}\left[\frac{\partial}{\partial x}\left((1+x/\rho)A_x\right) + \frac{\partial}{\partial y}\left((1+x/\rho)A_y\right) + \frac{\partial A_s}{\partial s}\right].$$

3.7.4 Laplacian of a Scalar

For a scalar $f(x, y, s)$, the Laplacian is

$$\nabla^2 f = \nabla \cdot (\nabla f).$$

Using the divergence formula above together with

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{s}}\frac{1}{h}\frac{\partial f}{\partial s},$$

we obtain

$$\nabla^2 f = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial f}{\partial s}\right)\right].$$

Thus,

$$\nabla^2 f = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial f}{\partial s}\right)\right]$$

This is the form commonly used in magnetic-field expansions in accelerator coordinates.

3.7.5 Curl Operator in Orthogonal Curvilinear Coordinates

Now we derive the curl explicitly.

Let

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_s\hat{\mathbf{s}}.$$

In general orthogonal curvilinear coordinates (u_1, u_2, u_3) with scale factors (h_1, h_2, h_3), the curl is

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.$$

For accelerator coordinates, identify

$$u_1 = x, \qquad u_2 = y, \qquad u_3 = s,$$

with scale factors

$$h_x = 1, \qquad h_y = 1, \qquad h_s = h.$$

So the curl becomes

$$\nabla \times \mathbf{A} = \frac{1}{h} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & h\hat{\mathbf{s}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial s} \\ A_x & A_y & hA_s \end{vmatrix}.$$

Now expand this determinant component by component.

3.7.6 x-Component of the Curl

The x-component is

$$(\nabla \times \mathbf{A})_x = \frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right].$$

Thus,

$$(\nabla \times \mathbf{A})_x = \frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right]$$

3.7.7 y-Component of the Curl

The y-component is

$$(\nabla \times \mathbf{A})_y = \frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right].$$

Thus,

$$(\nabla \times \mathbf{A})_y = \frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right]$$

3.7.8 s-Component of the Curl

The s-component is

$$(\nabla \times \mathbf{A})_s = \frac{1}{h}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right].$$

Thus,

$$(\nabla \times \mathbf{A})_s = \frac{1}{h}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

3.7.9 Final Curl Formula in (x, y, s) Coordinates

Combining the three components, we obtain

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\,\frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right] + \hat{\mathbf{y}}\,\frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right] + \hat{\mathbf{s}}\,\frac{1}{h}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

where

$$h = 1 + \frac{x}{\rho}.$$

So explicitly,

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\,\frac{1}{1+x/\rho}\left[\frac{\partial}{\partial y}\left(\left(1+\frac{x}{\rho}\right)A_s\right) - \frac{\partial A_y}{\partial s}\right] + \hat{\mathbf{y}}\,\frac{1}{1+x/\rho}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}\left(\left(1+\frac{x}{\rho}\right)A_s\right)\right] + \hat{\mathbf{s}}\,\frac{1}{1+x/\rho}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

This is the explicit curl operator in the standard planar s-dependent accelerator coordinates.

Magnetic Field Components Since

$$\mathbf{B} = \nabla \times \mathbf{A},$$

the field components are

Horizontal field

$$B_x = \frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right]$$

Vertical field

$$B_y = \frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right]$$

Longitudinal field

$$B_s = \frac{1}{h}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

These are the formulas most directly used when deriving canonical vector potentials for dipoles, quadrupoles, sextupoles, and octupoles in accelerator coordinates.

3.7.10 Simplification for Straight Coordinates

If the reference trajectory is straight, then

$$\rho \to \infty \qquad \Rightarrow \qquad h \to 1.$$

In that limit, the formulas reduce to the ordinary Cartesian ones:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{s}}\frac{\partial}{\partial s},$$

and

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x}\right) + \hat{\mathbf{s}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$

So the curved-coordinate formulas are a direct generalization of the Cartesian ones.

3.7.11 Common Gauge Simplification Used in Accelerator Hamiltonians

In many accelerator Hamiltonian derivations, one chooses a gauge such that

$$A_y = 0.$$

Then the curl simplifies to

$$B_x = \frac{1}{h}\frac{\partial}{\partial y}(hA_s),$$

$$B_y = \frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right],$$

$$B_s = -\frac{1}{h}\frac{\partial A_x}{\partial y}.$$

This is the form often used in canonical multipole derivations.

For static interior multipole fields with weak or no s-dependence, the dominant contribution often comes from A_s, while A_x carries fringe-field corrections.

3.7.12 Interpretation of the Scaled A_s

In accelerator notes, especially Hamiltonian derivations, one often defines the canonical longitudinal component with the metric factor already included. That is, instead of using the physical component $\hat{\mathbf{s}} \cdot \mathbf{A}$, one defines

$$A_s^{(\text{canonical})} = h\, \hat{\mathbf{s}} \cdot \mathbf{A}.$$

That convention is useful because the scale factor h appears naturally in both the Hamiltonian and the curl operator.

So when reading accelerator Hamiltonian notes, it is important to check whether A_s means:

- the physical s-component of $\mathbf{A}$, or
- the scaled canonical s-component including h.

The formulas above are written consistently with the standard orthogonal-coordinate representation where the curl contains hA_s.

3.7.13 Summary

For standard planar accelerator coordinates (x, y, s) with curvature $1/\rho$, define

$$h = 1 + \frac{x}{\rho}.$$

Then:

Gradient

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{s}}\frac{1}{h}\frac{\partial}{\partial s}$$

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{h}\left[\frac{\partial}{\partial x}(hA_x) + \frac{\partial}{\partial y}(hA_y) + \frac{\partial A_s}{\partial s}\right]$$

Scalar Laplacian

$$\nabla^2 f = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial f}{\partial s}\right)\right]$$

Curl

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\,\frac{1}{h}\left[\frac{\partial}{\partial y}(hA_s) - \frac{\partial A_y}{\partial s}\right] + \hat{\mathbf{y}}\,\frac{1}{h}\left[\frac{\partial A_x}{\partial s} - \frac{\partial}{\partial x}(hA_s)\right] + \hat{\mathbf{s}}\,\frac{1}{h}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

These are the differential operators needed for field expansions and Hamiltonian beam dynamics in curvilinear accelerator coordinates.

4 Hamiltonians of Magnetic Multipoles

4.1 Derivation of the s-Dependent Hamiltonian

4.1.1 Goal

In accelerator physics, it is often more convenient to use the path length along the reference trajectory, s, as the independent variable instead of time t.

The goal is to show why the Hamiltonian in this formulation is

$$H_s(x, p_x, y, p_y, t, -E; s) = -p_s.$$

This result follows from a canonical reparametrization of Hamiltonian mechanics.

4.1.2 Ordinary Hamiltonian Mechanics with Time as the Independent Variable

In ordinary Hamiltonian mechanics, the canonical coordinates are

$$(q_1, q_2, \dots, q_n),$$

with conjugate momenta

$$(p_1, p_2, \dots, p_n),$$

and the Hamiltonian is

$$H(q, p, t).$$

Hamilton's equations are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

For a relativistic charged particle in accelerator physics, one may begin with time t as the independent variable and write the canonical coordinates as

$$(x, y, s),$$

with conjugate momenta

$$(p_x, p_y, p_s).$$

Then the action is

$$S = \int \left(p_x \, dx + p_y \, dy + p_s \, ds - H \, dt \right).$$

This is the standard canonical one-form.

4.1.3 Extended Phase Space

To treat time on the same footing as the spatial coordinates, we introduce **extended phase space**.

In extended phase space, time t is treated as a canonical coordinate, and its conjugate momentum is

$$p_t = -H = -E.$$

So the extended coordinates are

$$(x, y, s, t),$$

with conjugate momenta

$$(p_x, p_y, p_s, p_t).$$

Since $p_t = -E$, we often write the extended variables as

$$(x, p_x, y, p_y, s, p_s, t, -E).$$

The canonical one-form becomes

$$p_x\,dx + p_y\,dy + p_s\,ds + p_t\,dt.$$

Using $p_t = -E$,

$$p_x\,dx + p_y\,dy + p_s\,ds - E\,dt.$$

Thus the action may be written as

$$S = \int \left(p_x\,dx + p_y\,dy + p_s\,ds - E\,dt\right).$$

At this stage, no specific independent variable has been chosen.

4.1.4 Reparametrization: Use s as the Independent Variable

Now we want to use s rather than t as the independent variable.

So we write everything as functions of s:

$$x = x(s), \qquad y = y(s), \qquad t = t(s).$$

Then

$$dx = x'\,ds, \qquad dy = y'\,ds, \qquad dt = t'\,ds,$$

where prime means derivative with respect to s:

$$x' = \frac{dx}{ds}, \qquad y' = \frac{dy}{ds}, \qquad t' = \frac{dt}{ds}.$$

Substituting into the action,

$$S = \int \left(p_x \frac{dx}{ds} + p_y \frac{dy}{ds} + p_s - E\frac{dt}{ds} \right) ds.$$

Rearrange:

$$S = \int \left[p_x x' + p_y y' - Et' + p_s \right] ds.$$

We want this to match the standard Hamiltonian form with s as the independent variable:

$$S = \int \left(p_x x' + p_y y' + p_t t' - H_s \right) ds.$$

Since $p_t = -E$, this becomes

$$S = \int \left(p_x x' + p_y y' - Et' - H_s \right) ds.$$

Comparing the two expressions, we identify

$$-H_s = p_s.$$

Therefore,

$$H_s = -p_s.$$

This is the desired result.

4.1.5 Why t Appears with Conjugate Momentum $-E$

A common point of confusion is why the pair is $(t, -E)$ rather than (t, E).

This comes from the canonical one-form in extended phase space:

$$\sum_i p_i \, dq_i - H \, dt.$$

If time is promoted to a canonical coordinate, then the coefficient of dt is the conjugate momentum of t. Since the term is $-H \, dt$, the conjugate momentum is

$$p_t = -H.$$

For a relativistic particle, the Hamiltonian is the total energy E, so

$$p_t = -E.$$

Thus the canonical pair is

$$(t, -E).$$

That is why the s-dependent Hamiltonian is naturally written as

$$H_s(x, p_x, y, p_y, t, -E; s).$$

4.1.6 Canonical Structure in the s-Dependent Formulation

Once s is chosen as the independent variable, the Hamiltonian is

$$H_s = -p_s.$$

The canonical variables are then

$$(x, p_x), \qquad (y, p_y), \qquad (t, -E).$$

Hamilton's equations with respect to s become

$$x' = \frac{\partial H_s}{\partial p_x}, \qquad p_x' = -\frac{\partial H_s}{\partial x},$$

$$y' = \frac{\partial H_s}{\partial p_y}, \qquad p_y' = -\frac{\partial H_s}{\partial y},$$

and for the longitudinal pair,

$$t' = \frac{\partial H_s}{\partial(-E)}, \qquad (-E)' = -\frac{\partial H_s}{\partial t}.$$

Equivalently,

$$E' = \frac{\partial H_s}{\partial t}.$$

So the s-dependent formulation is still a completely standard Hamiltonian system, just with a different independent variable.

4.1.7 Physical Interpretation

In a beamline or ring, particles progress through a sequence of elements arranged along the reference path. Because the machine itself is organized by position along that path, it is much more natural to describe the dynamics as evolution with respect to s rather than t.

In this picture:

- $x(s)$ and $y(s)$ describe transverse motion,
- $t(s)$ describes arrival time relative to the reference particle,
- $E(s)$ describes energy evolution,
- the generator of motion in s is the negative of the momentum conjugate to s, namely $-p_s$.

So the meaning of

$$H_s = -p_s$$

is completely analogous to the usual meaning of

$$H = -p_t.$$

It is simply the Hamiltonian appropriate to the chosen evolution parameter.

4.1.8 Compact Derivation from the Canonical One-Form

A short derivation is the following.

Start from the extended canonical one-form:

$$\Theta = p_x\, dx + p_y\, dy + p_s\, ds - E\, dt.$$

Now use s as the independent variable:

$$dx = x'ds, \qquad dy = y'ds, \qquad dt = t'ds.$$

Then

$$\Theta = \left(p_x x' + p_y y' - Et' + p_s\right) ds.$$

Compare with the standard canonical form for evolution in s:

$$\Theta = \left(p_x x' + p_y y' - Et' - H_s\right) ds.$$

Hence,

$$H_s = -p_s.$$

4.1.9 Final Result

Therefore, when s is chosen as the independent variable, the Hamiltonian is the negative of the canonical momentum conjugate to s:

$$H_s(x, p_x, y, p_y, t, -E; s) = -p_s.$$

This is the fundamental starting point for the s-dependent Hamiltonian formulation of accelerator beam dynamics.

4.2 s-Dependent Hamiltonian for Magnetic Elements

With s as the independent variable, the Hamiltonian is the negative of the canonical momentum conjugate to s,

$$H_s(x, p_x, y, p_y, t, -E; s) = -p_s.$$

In this formulation, the complete Hamiltonian is

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) P_s,$$

where

$$P_s = \left[\left(\frac{E - q\phi}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2\right]^{1/2}.$$

For static magnetic elements, $\phi = 0$. With the convenient gauge choice $A_y = 0$, a normalized transverse Hamiltonian is

$$H_{sT} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\left[1 - \left(p_x - \frac{qA_x}{p_0}\right)^2 - p_y^2\right]^{1/2},$$

where p_0 is the reference mechanical momentum.

This is the form into which the multipole vector potentials are inserted. We then derives canonical vector potentials for bend, quadrupole, skew quadrupole, sextupole, and octupole elements by first constructing the corresponding scalar potentials from Laplace's equation.

4.3 Derivation of the s-Dependent Hamiltonian for Magnetic Elements

4.4 Start from the Extended Canonical Form

In ordinary Hamiltonian mechanics with time as the independent variable, the action has the canonical form

$$S = \int \left(p_x \, dx + p_y \, dy + p_s \, ds - H \, dt \right).$$

If we move to **extended phase space**, then time becomes a canonical coordinate and its conjugate momentum is

$$p_t = -H = -E.$$

So the canonical one-form becomes

$$\Theta = p_x \, dx + p_y \, dy + p_s \, ds - E \, dt.$$

Now choose s as the independent variable. Then

$$dx = x' ds, \qquad dy = y' ds, \qquad dt = t' ds,$$

and the action becomes

$$S = \int \left(p_x x' + p_y y' - E t' + p_s \right) ds.$$

We want this to match the standard Hamiltonian form with s as the evolution parameter:

$$S = \int \left(p_x x' + p_y y' - E t' - H_s \right) ds.$$

Comparing the two expressions gives

$$H_s = -p_s.$$

Therefore,

$$H_s(x, p_x, y, p_y, t, -E; s) = -p_s.$$

This is the basic structural reason for the s-dependent Hamiltonian. It is the negative of the canonical momentum conjugate to the chosen independent variable s.

4.4.1 Mechanical Momentum in Curvilinear Accelerator Coordinates

Now we derive the explicit form of p_s.

Let the reference trajectory lie in the horizontal plane with curvature $1/\rho$. In the standard planar accelerator coordinates (x, y, s), the metric factor is

$$h = 1 + \frac{x}{\rho}.$$

So the line element is

$$d\ell^2 = dx^2 + dy^2 + h^2 ds^2.$$

Therefore the physical velocity squared is

$$v^2 = \dot{x}^2 + \dot{y}^2 + h^2 \dot{s}^2.$$

The mechanical momentum is

$$\Pi = \gamma m \mathbf{v}.$$

Its components along the local basis are

$$\Pi_x = \gamma m \dot{x}, \qquad \Pi_y = \gamma m \dot{y}, \qquad \Pi_s = \gamma m h \dot{s}.$$

Because of the geometric scale factor, the longitudinal component differs from the Cartesian form by the factor h.

The relativistic energy-momentum relation is then

$$E_{\text{mech}}^2 = m^2 c^4 + c^2 \left(\Pi_x^2 + \Pi_y^2 + \Pi_s^2 \right).$$

Here the mechanical total energy in electromagnetic fields is

$$E_{\text{mech}} = E - q\phi,$$

because the canonical energy variable E includes the scalar-potential contribution. Thus,

$$(E - q\phi)^2 = m^2 c^4 + c^2 \left(\Pi_x^2 + \Pi_y^2 + \Pi_s^2 \right).$$

So

$$\Pi_s = \left[\left(\frac{E - q\phi}{c} \right)^2 - \Pi_x^2 - \Pi_y^2 - m^2 c^2 \right]^{1/2}.$$

4.4.2 Canonical Momenta and the Vector Potential

From the electromagnetic Lagrangian, the canonical momentum satisfies

$$p_i = \Pi_i + qA_i$$

for each coordinate direction.

Thus,

$$\Pi_x = p_x - qA_x, \qquad \Pi_y = p_y - qA_y.$$

For the longitudinal direction in curvilinear coordinates, Symon defines the canonical potential component A_s with the geometry already absorbed into the s-component convention, so the s-canonical momentum relation is used in the form that ultimately gives the Hamiltonian below.

Substituting the transverse canonical-mechanical relations into Π_s, we obtain

$$P_s \equiv \Pi_s = \left[\left(\frac{E - q\phi}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2\right]^{1/2}.$$

This is exactly the P_s appearing in the desired Hamiltonian.

4.4.3 Why p_s Contains Both A_s and the Metric Factor

The canonical momentum conjugate to s comes from the Lagrangian in curvilinear coordinates. The key point is that the longitudinal mechanical momentum along the reference direction is scaled by the metric factor h, while the electromagnetic coupling adds the canonical potential term qA_s.

Accordingly, Symon's exact s-dependent Hamiltonian is written as

$$H_s = -qA_s - (1 + \rho^{-1}x)P_s.$$

Since $H_s = -p_s$, this means

$$p_s = qA_s + \left(1 + \frac{x}{\rho}\right) P_s.$$

This has a simple interpretation:

- qA_s is the electromagnetic contribution to the canonical momentum,
- $\left(1 + \frac{x}{\rho}\right) P_s$ is the mechanical longitudinal momentum in the curved coordinate system.

Thus the complete s-dependent Hamiltonian is

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right)\left[\left(\frac{E - q\phi}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2\right]^{1/2}.$$

This is the exact form used as the starting point in the note.

4.4.4 Specialization to Pure Transverse Motion in Static Magnetic Elements

Now restrict to the case used for transverse magnetic optics.

For a static magnetic element,

$$\phi = 0.$$

So the Hamiltonian becomes

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) \left[\left(\frac{E}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2 \right]^{1/2}.$$

The note then defines the kinetic momentum magnitude p by

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}.$$

Equivalently,

$$p^2 = \left(\frac{E}{c}\right)^2 - m^2c^2.$$

Substitute this into the square root:

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) \left[p^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2\right]^{1/2}.$$

This is exactly the transverse Hamiltonian quoted in the note before rescaling.

4.4.5 Gauge Choice $A_y = 0$

A convenient gauge is chosen such that

$$A_y = 0.$$

Then

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) \left[p^2 - (p_x - qA_x)^2 - p_y^2\right]^{1/2}.$$

This is the explicit s-dependent Hamiltonian for static magnetic transverse motion in the gauge used in the note.

4.4.6 Rescaling the Canonical Momenta

For pure transverse motion in a given magnetic element, the kinetic momentum magnitude p is constant. The note then rescales the canonical transverse momenta by $1/p$, not by $1/p_0$ at this step:

$$p_x \to \frac{p_x}{p}, \qquad p_y \to \frac{p_y}{p}.$$

After this rescaling, the kinetic parts become directly related to the slopes dx/ds and dy/ds. To preserve Hamilton's equations, one must also rescale the Hamiltonian by $1/p$. The note states this explicitly.

Thus the rescaled transverse Hamiltonian is

$$H_{sT} = -\frac{qA_s}{p} - \left(1 + \frac{x}{\rho}\right)\left[1 - \left(p_x - \frac{qA_x}{p}\right)^2 - p_y^2\right]^{1/2}.$$

If one now evaluates this around the reference particle with momentum p_0, then in lecture notation one often writes

$$H_{sT} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\left[1 - \left(p_x - \frac{qA_x}{p_0}\right)^2 - p_y^2\right]^{1/2}.$$

That is the normalized transverse Hamiltonian form used for inserting dipole, quadrupole, sextupole, and octupole vector potentials.

4.4.7 Why This Is the Right Starting Point for Multipoles

This Hamiltonian is the natural place to insert multipole vector potentials because:

1. the geometry of the reference orbit is already included through
$$1 + \frac{x}{\rho},$$
2. the electromagnetic effects enter through the canonical potentials A_s and A_x,
3. after paraxial expansion, the familiar polynomial Hamiltonians of dipole, quadrupole, sextupole, and octupole fields emerge naturally.

That is why the note first derives scalar potentials from Laplace's equation, then constructs canonical vector potentials, and finally substitutes them into this s-dependent Hamiltonian.

4.4.8 Final Derived Forms

Starting from the extended canonical formalism and curvilinear accelerator coordinates, we obtain

$$H_s(x, p_x, y, p_y, t, -E; s) = -p_s.$$

The exact s-dependent Hamiltonian is

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) P_s,$$

with

$$P_s = \left[\left(\frac{E - q\phi}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m^2c^2\right]^{1/2}.$$

For static magnetic elements, $\phi = 0$, and with the gauge choice $A_y = 0$,

$$H_s = -qA_s - \left(1 + \frac{x}{\rho}\right) \left[p^2 - (p_x - qA_x)^2 - p_y^2\right]^{1/2},$$

where

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}.$$

After normalizing by the reference momentum p_0, the transverse Hamiltonian becomes

$$H_{sT} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right) \left[1 - \left(p_x - \frac{qA_x}{p_0}\right)^2 - p_y^2\right]^{1/2}.$$

This is the form into which the canonical multipole vector potentials are inserted.

4.5 Strategy for Deriving Multipole Hamiltonians

The canonical derivation proceeds in three steps:

1. prescribe the magnetic field near the reference orbit,
2. derive a magnetic scalar potential ψ satisfying Laplace's equation,
3. construct a convenient vector potential $\mathbf{A}$ and insert it into H_{sT}.

Because the field region is current-free inside the magnet gap,

$$\nabla \times \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$

so a magnetic scalar potential can be introduced:

$$\mathbf{B} = \nabla\psi, \qquad \nabla^2\psi = 0.$$

Symmetry determines the angular dependence of ψ, and the longitudinal field-strength function $B(s)$, $Q(s)$, $S(s)$, $O(s)$, and so on introduces fringe and end-field dependence.

4.6 Dipole Hamiltonian

4.6.1 Uniform vertical dipole field

Consider a normal dipole with the dominant interior field

$$\mathbf{B} = B_0\,\hat{\mathbf{y}}.$$

A convenient scalar potential is

$$\psi = B_0 x.$$

One convenient vector potential that generates this field is

$$A_s = -B_0 x, \qquad A_x = 0, \qquad A_y = 0.$$

Indeed, since in the straight interior region

$$B_y = -\frac{\partial A_s}{\partial x},$$

we obtain

$$B_y = B_0.$$

This is the leading hard-edge interior dipole potential.

4.6.2 Insert into the transverse Hamiltonian

Insert

$$A_s = -B_0 x, \qquad A_x = 0$$

into

$$H_{sT} = -\frac{qA_s}{p_0} - \left(1 + \frac{x}{\rho}\right)\sqrt{1 - p_x^2 - p_y^2}.$$

Then

$$H_{sT} = \frac{qB_0}{p_0}x - \left(1 + \frac{x}{\rho}\right)\sqrt{1 - p_x^2 - p_y^2}.$$

Now expand the square root to second order in the paraxial approximation:

$$\sqrt{1-u} \approx 1 - \frac{u}{2}, \qquad u = p_x^2 + p_y^2.$$

Thus

$$H_{sT} \approx \frac{qB_0}{p_0}x - \left(1 + \frac{x}{\rho}\right)\left(1 - \frac{p_x^2 + p_y^2}{2}\right).$$

Dropping the additive constant -1 and keeping only the leading transverse terms gives

$$H_D \approx \frac{1}{2}(p_x^2 + p_y^2) + \left(\frac{qB_0}{p_0} - \frac{1}{\rho}\right)x.$$

For the reference particle, the design curvature satisfies

$$\frac{1}{\rho} = \frac{qB_0}{p_0}.$$

Therefore the linear term in x cancels, and we obtain the leading transverse dipole Hamiltonian

$$H_D^{(2)} = \frac{1}{2}(p_x^2 + p_y^2).$$

This reflects the fact that the dipole field is already absorbed into the reference curvature. In other words:

- the dipole determines the reference orbit,
- the transverse Hamiltonian describes deviations about that orbit.

4.6.3 Physical interpretation

The purpose of the dipole is not primarily to focus, but to bend the beam and define the reference trajectory. Once the reference curvature is chosen consistently with the field,

$$\frac{1}{\rho} = \frac{qB_0}{p_0},$$

the remaining lowest-order transverse Hamiltonian is just the free transverse motion about the reference orbit.

A more complete sector-dipole Hamiltonian includes weak focusing and edge effects.

4.7 Quadrupole Hamiltonian

4.7.1 Field and potentials

For a normal quadrupole, Symon specifies the field near the reference orbit as

$$\mathbf{B} = \hat{\mathbf{y}}\, Q(s)x + \hat{\mathbf{x}}\, Q(s)y.$$

A corresponding scalar potential is

$$\psi_0(x, y, s) = Q(s)xy.$$

Including longitudinal variation and applying Laplace's equation gives, through the relevant order,

$$\psi = Q(s)xy - \frac{Q''(s)}{12}(x^3y + xy^3).$$

A convenient vector potential is then

$$\mathbf{A} = \hat{\mathbf{s}}\left[\frac{Q}{2}(y^2 - x^2) + \frac{Q''}{48}(x^4 - 6x^2y^2 - y^4)\right] - \hat{\mathbf{x}}\,\frac{Q'}{2}xy^2.$$

The leading interior term is

$$A_s^{(2)} = \frac{Q}{2}(y^2 - x^2).$$

4.7.2 Hamiltonian

Insert this into the normalized transverse Hamiltonian and expand to lowest nontrivial order. Neglecting higher-order fringe terms gives

$$H_Q^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) - \frac{q}{p_0}\frac{Q}{2}(y^2 - x^2).$$

Define the normalized quadrupole strength

$$k_1(s) = \frac{qQ(s)}{p_0}.$$

Then

$$H_Q^{(2)} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{k_1}{2}x^2 - \frac{k_1}{2}y^2.$$

Hamilton's equations give

$$x'' + k_1(s)x = 0, \qquad y'' - k_1(s)y = 0.$$

This is the standard linear focusing-defocusing behavior of a normal quadrupole.

4.8 Skew Quadrupole Hamiltonian

For a skew quadrupole, the field is specified as

$$\mathbf{B} = \hat{\mathbf{x}}\, Q_s(s)x - \hat{\mathbf{y}}\, Q_s(s)y.$$

The scalar potential becomes

$$\psi = \frac{Q_s}{2}(x^2 - y^2) - \frac{Q_s''}{24}(x^4 - y^4).$$

A corresponding vector potential is

$$\mathbf{A} = \hat{\mathbf{s}}\left[Q_s xy - \frac{Q_s''}{6}x^3 y\right] + \hat{\mathbf{x}}\, Q_s'\left(\frac{y^3}{6} - \frac{x^2 y}{2}\right).$$

The leading term is

$$A_s^{(2)} = Q_s xy.$$

Substituting into the Hamiltonian gives the quadratic skew-quadrupole Hamiltonian

$$H_{Qs}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) - \frac{q}{p_0}Q_s xy.$$

With

$$k_{1s}(s) = \frac{qQ_s(s)}{p_0},$$

we obtain

$$H_{Qs}^{(2)} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_{1s}xy.$$

This term couples the horizontal and vertical planes. Skew quadrupoles are therefore the canonical source of linear transverse coupling.

4.9 Sextupole Hamiltonian

Symon specifies a sextupole by the field near the reference orbit. The corresponding scalar potential is given as

$$\psi = \frac{1}{6}S(s)(3x^2 y - y^3).$$

The associated vector potential is

$$\mathbf{A} = \hat{\mathbf{s}}\, S\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right) + \hat{\mathbf{x}}\, S'\left(-\frac{1}{4}x^2y^2 + \frac{1}{24}y^4\right).$$

The dominant interior term is

$$A_s^{(3)} = S\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right).$$

Define

$$k_2(s) = \frac{qS(s)}{p_0}.$$

Then the leading sextupole Hamiltonian becomes

$$H_S = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{k_2}{6}x^3 - \frac{k_2}{2}xy^2.$$

This produces the familiar quadratic transverse kicks:

$$x'' = -\frac{k_2}{2}(x^2 - y^2), \qquad y'' = k_2 xy.$$

Thus the sextupole is the leading nonlinear magnetic element beyond the quadrupole and is essential for chromaticity correction.

4.10 Octupole Hamiltonian

For the octupole, Symon gives the scalar potential as

$$\psi = -O(s)(x^3y - xy^3).$$

The vector potential is

$$\mathbf{A} = \hat{\mathbf{s}}\, O\left(\frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4\right).$$

Define

$$k_3(s) = \frac{qO(s)}{p_0}.$$

Then the leading octupole Hamiltonian is

$$H_O = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_3\left(\frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4\right).$$

The octupole introduces cubic restoring forces and is the first common magnetic element whose leading effect is intrinsically quartic in the coordinates. It is used for amplitude-dependent tune shift and nonlinear detuning.

4.11 Unified Polynomial Hamiltonian Forms

At leading order, the transverse Hamiltonian may be written schematically as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y; s),$$

with

$$V_{\text{dipole}} = 0 \quad \text{after subtraction of the reference curvature,}$$

$$V_{\text{quad}} = \frac{k_1}{2}(x^2 - y^2),$$

$$V_{\text{skew}} = -k_{1s}xy,$$

$$V_{\text{sext}} = \frac{k_2}{6}(x^3 - 3xy^2),$$

$$V_{\text{oct}} = -\frac{k_3}{4}(x^4 - 6x^2y^2 + y^4).$$

These are the canonical multipole Hamiltonians used throughout accelerator beam dynamics. The value of Symon's derivation is that it shows how these familiar polynomial forms emerge from the full accelerator Hamiltonian with canonical vector potentials, including fringe-field corrections through $B'(s)$, $Q'(s)$, $Q''(s)$, $S'(s)$, and related terms.

4.12 Simple Examples and Physical Interpretation

4.12.1 Free particle in one dimension

For a free nonrelativistic particle,

$$H = \frac{p^2}{2m}.$$

Hamilton's equations give

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = 0.$$

So the momentum is constant and the particle moves at constant velocity.

4.12.2 Harmonic oscillator

For

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

Hamilton's equations give

$$\dot{x} = \frac{p}{m}, \qquad \dot{p} = -kx,$$

which combine to give

$$m\ddot{x} + kx = 0.$$

4.12.3 Multipoles as Hamiltonian potentials

The magnetic multipole results above show a powerful physical principle:

- a dipole defines the reference curvature,
- a quadrupole acts like a quadratic Hamiltonian potential,
- a sextupole acts like a cubic Hamiltonian potential,
- an octupole acts like a quartic Hamiltonian potential.

Thus accelerator magnets are naturally understood as polynomial generators in the Hamiltonian.

4.13 Toward Linear Beam Optics

Starting from the Hamiltonian framework, the next steps are:

1. express motion in curvilinear coordinates,
2. expand around the reference trajectory,
3. apply the paraxial approximation,
4. linearize the equations of motion,
5. derive the transverse focusing equations.

The result will be equations of the form

$$x'' + K_x(s)x = 0, \qquad y'' + K_y(s)y = 0,$$

which are the starting point of linear transverse optics.

Thus the Hamiltonian formalism developed here is the direct gateway to:

- Hill's equation,
- transfer matrices,
- Twiss parameters,
- emittance,
- longitudinal phase-space dynamics,
- nonlinear beam dynamics.

5 Summary

In this lecture, we introduced the Hamiltonian foundation of beam dynamics and extended it to the canonical derivation of magnetic multipoles.

1) Beam dynamics begins by asking how charged particles move through accelerator fields. 2) Hamiltonian mechanics is preferred because it uses canonical variables and preserves symplectic structure. 3) The relativistic Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\mathbf{A}\cdot\mathbf{v} - q\phi.$$

4) The canonical momentum is

$$\mathbf{p} = \gamma m\mathbf{v} + q\mathbf{A}.$$

5) The relativistic Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}, t) = \sqrt{m^2c^4 + c^2(\mathbf{p} - q\mathbf{A})^2} + q\phi.$$

6) Accelerator motion is described relative to a reference particle and reference trajectory. 7) Curvilinear coordinates and the paraxial approximation make beam optics possible. 8) Symplecticity is a central structural property of beam dynamics. 9) Magnetic lattice elements enter the Hamiltonian through canonical vector potentials. 10) The leading multipole Hamiltonians are polynomial in the transverse coordinates: - dipole: reference curvature, - quadrupole: quadratic, - skew quadrupole: bilinear coupling, - sextupole: cubic, - octupole: quartic.

References

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