

2. Electromagnetic Fields in Accelerator Physics

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1 Magnetic Field Expansion and General Magnetic Multipole Devices

1.1 Introduction

In accelerator physics, magnetic fields are the primary tools for bending, guiding, and focusing charged-particle beams. Dipole magnets provide beam steering, quadrupoles provide linear focusing, and higher-order multipoles are used to correct aberrations or to tailor nonlinear beam dynamics.

The mathematical foundation is provided by Maxwell’s equations in source-free regions, together with scalar or vector potentials satisfying Laplace’s equation, $\nabla^2 \mathbf{B} = 0$. In this note, we focus on the magnetic-field expansion and on the interpretation of general magnetic multipole devices.

1.2 Magnetostatic Fields in a Source-Free Region

In a static, charge-free, and current-free beam region, Maxwell’s equations reduce to

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0.$$

Therefore, the magnetic field can be expressed in terms of a potential. For transverse-field problems, it is convenient to introduce the complex variable

$$z = x + iy,$$

and a complex potential

$$P(z) = A_z(z) + iV(z),$$

where A_z is the longitudinal component of the vector potential and V is a scalar-potential-like function used to represent the transverse field compactly.

The complex magnetic field is written as

$$B \equiv B_x - iB_y = i \frac{\partial P}{\partial z}.$$

The transverse field components may also be expressed as

$$B_x = \frac{\partial V}{\partial x}, \quad B_y = -\frac{\partial V}{\partial y},$$

or equivalently in terms of A_z ,

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}.$$

Thus the transverse magnetic field is generated from an analytic function, and the analyticity condition guarantees consistency with Laplace's equation.

1.3 Pure Multipole Field Expansion

1.3.1 Complex power-series expansion

Because $P(z)$ is analytic, it can be expanded as a power series:

$$P(z) = \sum_{n \geq 0} C_n z^n = \sum_{n \geq 0} C_n (x + iy)^n = \sum_{n \geq 0} C_n r^n e^{in\phi},$$

where the coefficients C_n are in general complex:

$$C_n = \lambda_n + i\mu_n.$$

The real parts λ_n correspond to **upright multipoles**, while the imaginary parts μ_n correspond to **skew multipoles**.

For upright multipoles, the horizontal midplane ($y = 0$) is a symmetry plane, and therefore

$$B_x(x, 0) = 0, \quad B_y(x, 0) \neq 0.$$

This midplane symmetry is important in accelerator beam dynamics because it simplifies the transverse motion and avoids transverse coupling in the ideal case.

1.3.2 Multipole order

Each power of z corresponds to a particular multipole order:

- $n = 1$: dipole
- $n = 2$: quadrupole
- $n = 3$: sextupole
- $n = 4$: octupole

- $n = 5$: decapole

In practice, a real magnet is designed so that one multipole dominates while the unwanted higher-order components are minimized.

1.3.3 Field Gradients and Normalized Strength Parameters

The multipole coefficients are related to physical field gradients. In accelerator physics, it is convenient to define normalized, energy-independent strength parameters using the beam rigidity $B\rho$:

$$B\rho = \frac{p_0}{e} = \frac{\beta E}{ce}.$$

For magnetic fields, one often writes

$$(B\rho)_m = \frac{\beta E(\text{GV})}{0.29979}.$$

More in general,

$$B\rho = \frac{p}{q}.$$

The normalized multipole strengths are denoted by

$$S_n = \frac{ec}{\beta E} s_n.$$

For the low-order magnets, common notation is:

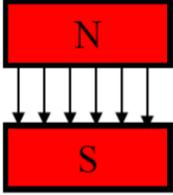
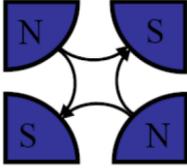
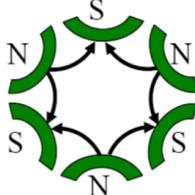
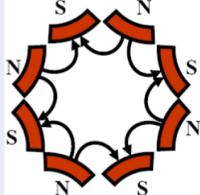
- dipole: ρ^{-1}
- quadrupole: k
- sextupole: m
- octupole: r

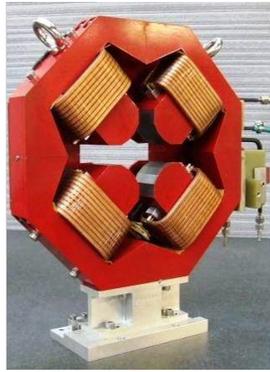
Then the midplane vertical magnetic field can be expanded as

$$\frac{e}{p_0} B_y = \frac{1}{\rho_0} + kx + \frac{1}{2}mx^2 + \frac{1}{6}rx^3 + \dots.$$

This is one of the most useful field expansions in beam dynamics.

1.4 Multipoles

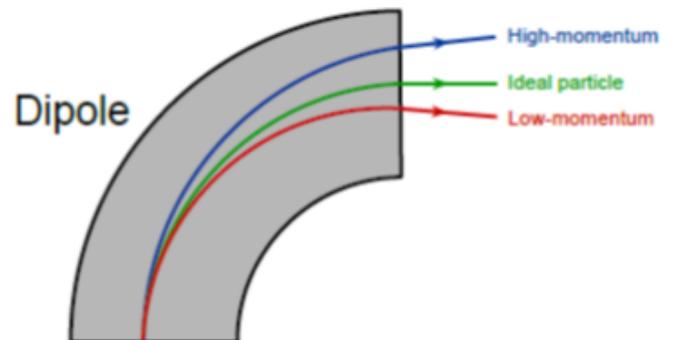
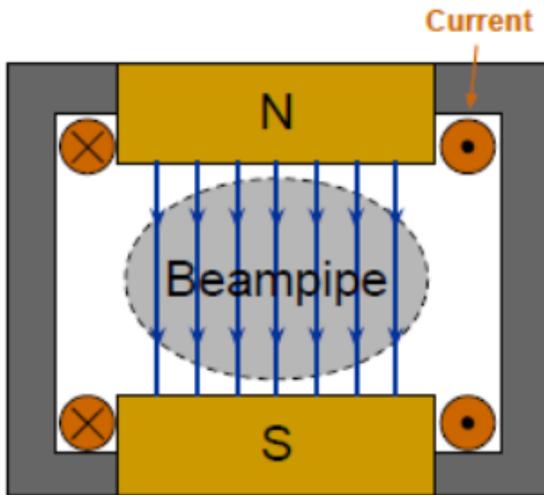
n = 0	n = 1	n = 2	n = 3	n = 4
Drift	Dipole	Quadrupole	Sextupole	Octupole
Beam pipe, etc.				
$B_x = 0$	$B_x = 0$	$B_x = B'y$	$B_x = B''xy$	$B_x = \frac{1}{6}B'(3x^2y - y^3)$
$B_y = 0$	$B_y = B_0$	$B_y = B'x$	$B_y = \frac{1}{2}B''(x^2 - y^2)$	$B_y = \frac{1}{6}B''(x^3 - 3xy^2)$



1.4.1 Dipole

$$\frac{e}{p_0} B_x = 0, \quad \frac{e}{p_0} B_y = \frac{1}{\rho_0}.$$

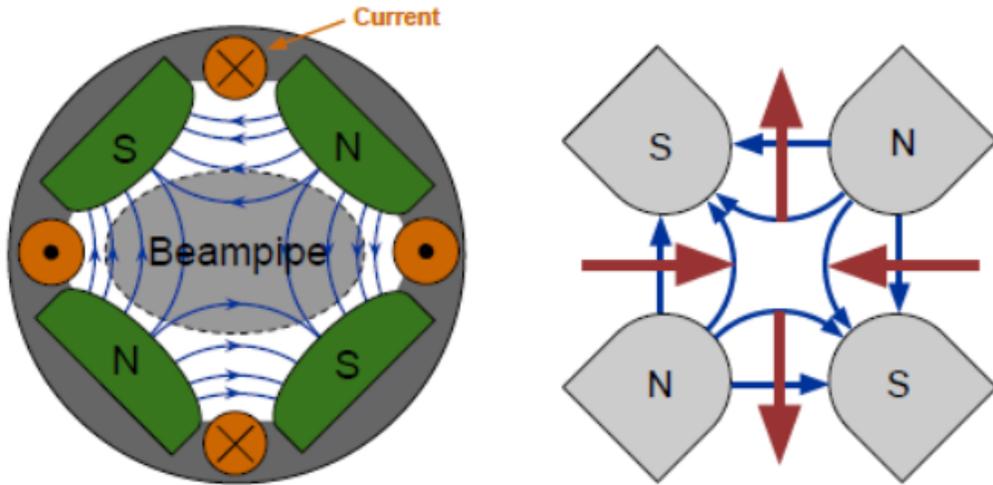
A dipole produces a nearly uniform transverse field and bends the beam trajectory.



1.4.2 Quadrupole

$$\frac{e}{p_0} B_x = ky, \quad \frac{e}{p_0} B_y = kx.$$

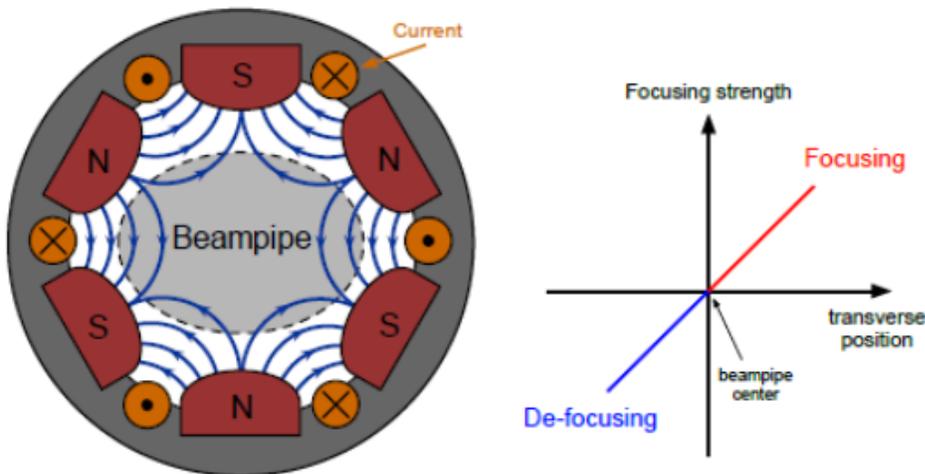
A quadrupole provides linear focusing in one transverse plane and linear defocusing in the other.



1.4.3 Sextupole

$$\frac{e}{p_0} B_x = mxy, \quad \frac{e}{p_0} B_y = \frac{1}{2}m(x^2 - y^2).$$

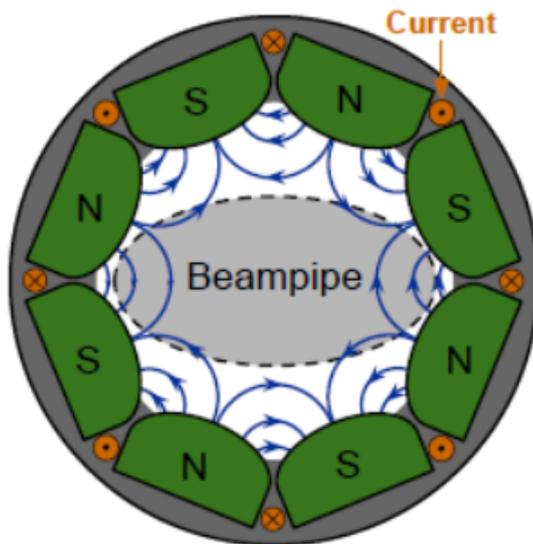
A sextupole is a nonlinear focusing element and is commonly used for chromaticity correction.



1.4.4 Octupole

$$\frac{e}{p_0} B_x = \frac{1}{6}r(3x^2y - y^3), \quad \frac{e}{p_0} B_y = \frac{1}{6}r(x^3 - 3xy^2).$$

An octupole introduces higher-order nonlinear effects such as amplitude-dependent tune shifts.



1.4.5 Skew multipoles

A skew multipole is a rotated version of the corresponding upright multipole.

Examples:

- skew quadrupole = upright quadrupole rotated by 45°
- skew sextupole = upright sextupole rotated by 30°

Skew multipoles generally couple the horizontal and vertical beam motion. In real accelerators, they may arise from alignment errors or may be intentionally introduced for correction purposes.

1.5 General Midplane Expansion of the Magnetic Field

For upright multipoles in the horizontal midplane ($y = 0$), the horizontal field vanishes:

$$B_x = 0.$$

Then the vertical field can be written as a one-dimensional expansion around the reference orbit:

$$\frac{e}{p_0} B_y = \frac{1}{\rho_0} + kx + \frac{1}{2}mx^2 + \frac{1}{6}rx^3 + \dots + \frac{1}{(n-1)!} S_n x^{n-1}.$$

This expansion is used extensively in beam optics and tracking. It describes how dipole, quadrupole, sextupole, and higher-order components contribute to the magnetic environment near the beam axis.

1.6 General Magnetic Multipole Devices

1.6.1 Dipole magnets

A dipole magnet corresponds to the ($n = 1$) multipole. It produces an approximately uniform transverse magnetic field. In an upright dipole, the field is vertical and the beam bends in the horizontal plane.

The dipole determines the reference curvature of the beam trajectory.

1.6.2 Quadrupole magnets

A quadrupole corresponds to the ($n = 2$) multipole and provides linear focusing. Its ideal pole profile is hyperbolic.

For an upright quadrupole, the pole surfaces satisfy

$$xy = \text{const.}$$

For the rotated quadrupole, the profile is described by

$$x^2 - y^2 = \text{const.}$$

An upright quadrupole focuses in one plane and defocuses in the other. This is the fundamental focusing element in accelerator lattices.

1.6.3 Combined-function magnets

A magnet containing both dipole and quadrupole components is called a **combined-function magnet** or **synchrotron magnet**.

A simple interpretation is that it is equivalent to a quadrupole displaced transversely from the beam axis. Then the field may be written as

$$B_y = g(x - x_0) = gx - gx_0.$$

The constant term gives a dipole field, while the linear term gives a quadrupole gradient.

Thus, the same magnet can simultaneously bend and focus the beam.

1.6.4 Sextupole magnets

A sextupole corresponds to the ($n = 3$) multipole. Its field is quadratic in the transverse coordinates:

$$\frac{e}{p_0} B_x = mxy, \quad \frac{e}{p_0} B_y = \frac{1}{2}m(x^2 - y^2).$$

Sextupoles are primarily used to correct chromatic aberrations in accelerator lattices. Since particles with different momenta experience different quadrupole focusing, sextupoles are introduced to compensate for this momentum dependence.

One sextupole pole profile may be described by

$$3x^2y - y^3 = R^3,$$

and the full magnet geometry is obtained by rotation through 60°

1.6.5 Octupole and higher-order multipoles

Octupoles and higher-order multipoles are important in nonlinear beam dynamics. They are used in applications such as:

- dynamic aperture control,
- resonance compensation,
- amplitude-dependent tune manipulation,
- beam halo control.

In general, the n -th multipole potential is written as

$$P_n(x, y) = C_n(x + iy)^n.$$

Odd-order upright multipoles such as dipoles, sextupoles, and decapoles have a central pole on the vertical axis, while even-order multipoles such as quadrupoles and octupoles do not.

The full magnet geometry is obtained from one pole profile by rotational symmetry.

1.7 Toward the General Transverse Magnetic-Field Expansion

The pure multipole treatment assumes:

- purely transverse fields,
- straight magnets,
- Cartesian coordinates,
- no curvature effects.

This is sufficient for many introductory beam-dynamics problems, but not for high-precision accelerator modeling.

A more general treatment uses a curvilinear coordinate system that follows the reference trajectory. In that case, the magnetic scalar potential must satisfy Laplace's equation in curvilinear coordinates, and the field expansion contains not only the pure multipole terms but also additional geometric or kinematic terms.

1.8 General Transverse Magnetic-Field Expansion in Curvilinear Coordinates

Let x and y be transverse coordinates relative to a reference trajectory with curvatures κ_x and κ_y . The metric factor is

$$h = 1 + \kappa_x x + \kappa_y y.$$

Then Laplace's equation becomes

$$\nabla^2 V = \frac{1}{h} \left[\frac{\partial}{\partial x} \left(h \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{1}{h} \frac{\partial V}{\partial z} \right) \right] = 0.$$

The scalar potential is expanded as

$$\frac{ec}{\beta E} V(x, y, z) = \sum_{p, q \geq 0} A_{pq}(z) \frac{x^p y^q}{p! q!}.$$

Substituting this series into Laplace's equation gives recursion relations for the coefficients A_{pq} . These recursion relations generate:

1. the usual pure multipole terms,
2. additional curvature-dependent and longitudinally varying terms.

These additional terms are often called **kinematic terms**.

1.9 Pure Multipole Content Within the General Expansion

Because Laplace's equation is second order, the lowest-order coefficients must be specified explicitly.

The leading coefficients are

$$A_{10} = -\kappa_y, \quad A_{01} = \kappa_x,$$

and the quadratic coefficients are associated with quadrupole strengths:

$$A_{20} = k, \quad A_{11} = \underline{k}, \quad A_{02} = -k.$$

For the pure multipole contribution, the recursion simplifies to

$$A_{p+2, q; n} + A_{p, q+2; n} = 0, \quad p + q + 2 \leq n.$$

This reproduces the familiar polynomial multipole potentials.

1.9.1 Examples up to fifth order

Dipole

$$-\frac{e}{p_0} V_1 = -\kappa_y x + \kappa_x y$$

Quadrupole

$$-\frac{e}{p_0} V_2 = -\frac{1}{2} k (x^2 - y^2) + \underline{k} xy$$

Sextupole

$$-\frac{e}{p_0} V_3 = -\frac{1}{6} m (x^3 - 3xy^2) + \frac{1}{6} \underline{m} (3x^2y - y^3)$$

Octupole

$$-\frac{e}{p_0}V_4 = -\frac{1}{24}r(x^4 - 6x^2y^2 + y^4) + \frac{1}{24}r(x^3y - xy^3)$$

Decapole

$$-\frac{e}{p_0}V_5 = -\frac{1}{120}d(x^5 - 10x^3y^2 + 5xy^4) + \frac{1}{120}d(5x^4y - 10x^2y^3 + y^5).$$

Thus the familiar pure multipole fields appear naturally within the more general field expansion.

1.10 Kinematic Terms

The general expansion contains more than the pure multipoles. Even if a magnet is nominally a dipole or quadrupole, a realistic curvilinear coordinate system and longitudinal field variation introduce extra terms.

These kinematic terms depend on quantities such as:

- the reference curvature (κ_x, κ_y) ,
- longitudinal derivatives of field strengths such as k' and k'' ,
- products of lower-order multipole strengths.

Physically, these terms represent effects such as:

- fringe fields,
- finite magnet width,
- curvature corrections,
- mixed geometric contributions.

Thus the complete field seen by the beam is generally richer than the idealized pure multipole model.

1.11 Interpretation for Accelerator Physics

The main lesson is:

- **pure multipoles** describe the intended design field,
- **kinematic terms** describe geometric and longitudinal corrections required by Maxwell's equations in realistic accelerator coordinates.

For many first-order beam-optics calculations, the pure multipole model is sufficient. However, for precision optics, nonlinear dynamics, and realistic magnet modeling, the general field expansion becomes essential.

1.12 Summary

1. In source-free regions, transverse magnetic fields can be represented by an analytic complex potential.
2. Expanding the potential in powers of $z = x + iy$ yields the standard multipole series.
3. Upright and skew multipoles are represented by the real and imaginary parts of the coefficients.
4. Dipole, quadrupole, sextupole, octupole, and higher-order fields arise as successive polynomial orders.

5. Normalized strengths such as ρ^{-1} , k , m , and r are defined using beam rigidity.
6. Real magnetic devices realize these field patterns through characteristic pole geometries.
7. Misalignment produces skew components and spill-down.
8. The general transverse magnetic-field expansion in curvilinear coordinates includes both pure multipole and kinematic terms.

2 RF Waveguides, Cavities, and RF Parameters

2.1 Introduction

Charged particles are accelerated by radio-frequency (RF) electromagnetic fields. For efficient acceleration, the fields must provide a **longitudinal electric field component** along the beam direction and must be synchronized with the particle motion. In accelerator applications, this is achieved either by using traveling-wave structures with suitable phase velocity, or by using standing-wave fields in resonant cavities placed at specific locations along the beam path.

This note focuses on the accelerator-physics interpretations of waveguides, resonant cavities, transit-time effects, cavity losses, shunt impedance, quality factor, and the practical choice of RF parameters for circular accelerators.

2.2 RF Waveguides and Cavities

2.2.1 Motivation

A free plane electromagnetic wave is not suitable for direct charged-particle acceleration because it does not provide the required longitudinal electric field component in the propagation direction. To obtain such a component, one must impose appropriate boundary conditions by surrounding the field with conducting surfaces, thereby creating waveguide or cavity modes.

Two general acceleration mechanisms are emphasized:

1. A **traveling wave** whose phase velocity matches the particle velocity.
2. A **resonant cavity field** where the particle gains net positive energy while traversing the cavity, provided the transit time is not too long compared with the RF period.

2.2.2 Wave Equation in a Charge-Free Region

In a charge-free medium, Maxwell's equations can be written as

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt},$$

$$\nabla \cdot \mathbf{B} = 0, \quad c^2 \nabla \times \mathbf{B} = \frac{d\mathbf{E}}{dt}.$$

Assuming time-harmonic fields of the form

$$U = U_0 e^{i\omega t}, \quad U = E \text{ or } B,$$

the wave equations become

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0, \quad \nabla^2 \mathbf{B} + k^2 \mathbf{B} = 0,$$

with

$$k = \sqrt{\mu\epsilon} \frac{\omega}{c}.$$

For a free plane wave traveling in the z -direction, the solution is

$$E = E_0 e^{i(\omega t - kz)},$$

and the phase velocity is

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\epsilon}}.$$

However, such a plane wave still lacks the longitudinal field component needed for sustained particle acceleration.

2.2.3 Rectangular Waveguide Modes

Consider a rectangular metallic waveguide aligned along the z -axis. To obtain a nonzero longitudinal electric field, one seeks solutions for E_z that satisfy boundary conditions at the conducting walls. Separation of variables leads to a mode structure characterized by transverse wave numbers k_x and k_y , with

$$k_c^2 = k_x^2 + k_y^2,$$

and

$$k_z^2 = k^2 - k_c^2.$$

The longitudinal electric field can be written as

$$E_z = E_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{i(\omega t - k_z z)},$$

where a and b are the waveguide dimensions and (m, n) are integers specifying the transverse mode.

The phase velocity is

$$v_{\text{ph}} = \frac{\omega}{k_z} = \frac{c}{\sqrt{1 - (k_c/k)^2}},$$

which is always greater than c when the mode propagates in vacuum. Therefore, a smooth rectangular waveguide cannot directly accelerate particles efficiently, because the wave phase outruns any material particle.

Waveguide modes are classified into:

- **TM modes:** $B_z = 0$, with nonzero E_z
- **TE modes:** $E_z = 0$, with nonzero B_z

Only **TM modes** are useful for acceleration, because they provide a longitudinal electric field. In a rectangular guide, the lowest accelerating TM mode is the **TM₁₁** mode.

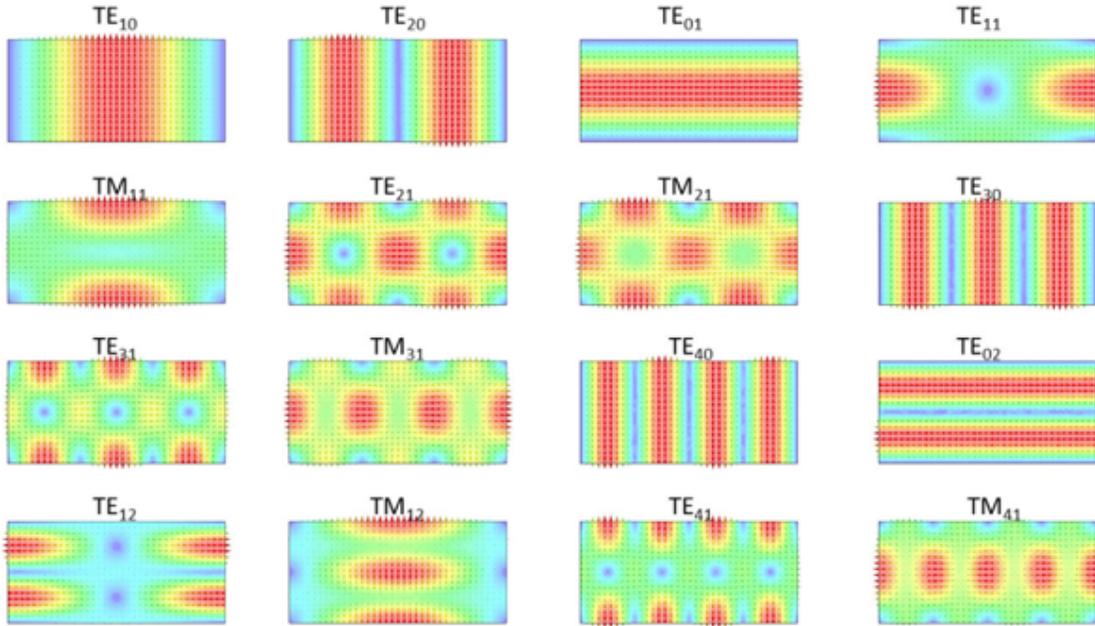
The waveguide wavelength is

$$\lambda_z = \frac{2\pi}{\sqrt{k^2 - k_c^2}},$$

which is always longer than the free-space wavelength. The group velocity is

$$v_g = \frac{d\omega}{dk_z} = c\sqrt{1 - \frac{k_c^2}{k^2}} < c.$$

Thus the phase velocity exceeds c , while the group velocity remains below c , as expected for energy transport.



2.2.4 Cylindrical Waveguide Modes

Because accelerator cavities are more naturally built with cylindrical symmetry and tighter mechanical tolerances, cylindrical waveguides and cavities are especially important. In cylindrical coordinates, the wave equation for E_z becomes a Bessel equation, and the longitudinal electric field takes the form

$$E_z = E_0 J_m(k_c r) e^{i(\omega t - m\phi - k_z z)}.$$

The conducting boundary at radius a requires that $E_z = 0$ there, so the allowed modes are determined by the zeros of the Bessel function J_m . For the lowest-order case $m = 0$, the first zero is

$$k_c a = 2.405.$$

This gives the familiar TM_{010} mode as the fundamental cylindrical cavity mode.

Mode integers m , n , and p are defined as

$$k_z = \frac{p\pi}{l}$$

and

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2.$$

For a cylindrical cavity with end caps forming a standing wave, the resonance frequency of the TM_{010} mode is

$$\omega_{010} = c\sqrt{\mu\epsilon} \frac{2.405}{a}.$$

The corresponding longitudinal electric field is

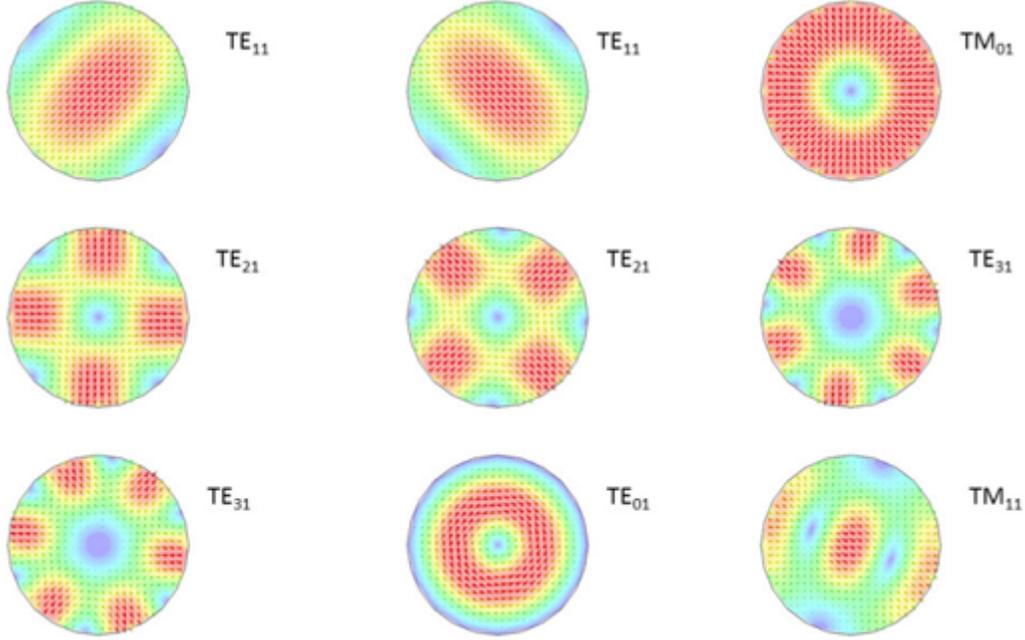
$$E_z = E_{z,010} J_0\left(\frac{2.405 r}{a}\right) e^{i\omega_{010} t}.$$

This mode is especially important because it has an axial accelerating electric field and no dependence on z for the fundamental $p = 0$ mode.

As in rectangular waveguides, traveling-wave cylindrical modes have phase velocity greater than c , so further structural modification or use of standing waves is required for acceleration. The group velocity again satisfies

$$v_g = \frac{c^2}{v_{\text{ph}}} < c.$$

The full TM-mode field components can be derived from Maxwell's equations once E_z is known. For the TM_{010} mode, the dominant transverse field is the azimuthal magnetic field B_ϕ , while the transverse electric fields vanish for the fundamental mode.



2.3 RF Cavities

2.3.1 Standing-Wave Cavities

A practical accelerating cavity is obtained by superposing two waves traveling in opposite directions, thereby creating a **standing wave**. In a rectangular cavity, this leads to

$$E_z = 2E_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \cos \frac{p\pi z}{d} e^{i\omega t},$$

where p specifies the longitudinal standing-wave structure. The corresponding resonant modes are labeled TM_{mnp} . The lowest rectangular cavity mode is TM_{110} .

In practice, however, **cylindrical cavities** are preferred for accelerating systems, because they are easier to fabricate accurately and support the fundamental TM_{010} mode very naturally. With end caps added to a cylindrical guide, the TM_{010} mode becomes a standing-wave cavity mode with resonance frequency

$$\omega_{010} = c\sqrt{\mu\epsilon} \frac{2.405}{a}.$$

The electric field on axis is then

$$E_z = 2E_{z,010} J_0 \left(\frac{2.405 r}{a} \right) \cos(\omega_{010} t).$$

This is the fundamental accelerating mode of a pillbox cavity.

The cavity radius is inversely proportional to resonance frequency, so higher RF frequencies allow smaller cavities. Typical values are about 3 GHz for electron linacs and about 500 MHz for storage rings, with corresponding cavity radii of a few centimeters to a few tens of centimeters.

2.3.2 Energy Gain and Transit-Time Factor

When a charged particle traverses an RF accelerating gap, the electric field is time-dependent. Even if the peak field is large, the particle does not generally see that peak field during the entire transit through the gap. As a result, the actual energy gain is smaller than the naive value obtained by multiplying the peak field by the gap length.

This reduction is described by the **transit-time factor**.

The transit-time factor is one of the most important concepts in RF acceleration, especially for low- and medium-energy particles, where the particle velocity is not yet close to the speed of light.

Definition of Energy Gain Consider a particle with charge q traversing an accelerating gap of length d along the z -axis.

The energy gain is

$$\Delta W = q \int_{z_i}^{z_f} E_z(z, t) dz,$$

where E_z is the longitudinal electric field experienced by the particle.

If the field were static and uniform, one would simply obtain

$$\Delta W = qE_0d.$$

However, in an RF cavity the field oscillates in time, so the particle sees a field that changes while it crosses the gap.

Simple Model of the RF Gap Field Assume the electric field in the gap is spatially uniform and sinusoidal in time:

$$E_z(t) = E_0 \cos(\omega t + \phi).$$

Let the particle enter the gap at $z = 0$ and exit at $z = d$.

Assume the particle velocity is approximately constant across the gap:

$$v = \text{constant}.$$

Then the particle position is

$$z = vt,$$

or equivalently,

$$t = \frac{z}{v}.$$

Substituting this into the RF field gives the field seen by the particle as a function of position:

$$E_z(z) = E_0 \cos\left(\omega \frac{z}{v} + \phi\right).$$

Therefore, the energy gain becomes

$$\Delta W = q \int_0^d E_0 \cos\left(\omega \frac{z}{v} + \phi\right) dz = qE_0 d \frac{\sin(\omega d/2v)}{\omega d/2v} \cos\left(\phi + \frac{\omega d}{2v}\right).$$

Definition of the Transit-Time Factor Define the peak gap voltage as

$$V_0 = E_0 d.$$

Then the energy gain becomes

$$\Delta W = qV_0 \frac{\sin(\omega d/2v)}{\omega d/2v} \cos\left(\phi + \frac{\omega d}{2v}\right).$$

The dimensionless factor

$$T = \frac{\sin(\omega d/2v)}{\omega d/2v}$$

is called the **transit-time factor**.

Thus,

$$\Delta W = qV_0 T \cos\left(\phi + \frac{\omega d}{2v}\right)$$

or equivalently, if the phase is referenced to the gap center,

$$\Delta W = qV_0 T \cos \phi_s$$

where ϕ_s is the phase of the particle relative to the RF field at the gap center.

Physical Meaning The transit-time factor compares:

- the **actual energy gain** in a time-varying RF field,
- to the **ideal energy gain** if the full peak voltage acted throughout the gap.

So,

$$T = \frac{\text{actual energy gain at optimal phase}}{qV_0}$$

Since the field changes while the particle is in transit, one always has

$$0 < T \leq 1.$$

- $T = 1$: ideal case, no reduction
- $T < 1$: realistic case, finite crossing time reduces the net gain

Interpretation of the Argument The quantity

$$\frac{\omega d}{2v}$$

is half the RF phase advance seen by the particle while crossing the gap.

Using

$$\omega = 2\pi f, \quad \lambda = \frac{2\pi c}{\omega},$$

we can also write

$$\frac{\omega d}{2v} = \frac{\pi d}{\beta\lambda},$$

where

$$\beta = \frac{v}{c}.$$

Therefore,

$$T = \frac{\sin(\pi d/\beta\lambda)}{\pi d/\beta\lambda}$$

This form is widely used in accelerator physics.

Pillbox Cavity Example For a simple pillbox cavity, one often takes the cavity length as

$$d = \frac{\lambda}{2}.$$

For an ultra-relativistic particle,

$$v \approx c \quad \Rightarrow \quad \beta \approx 1.$$

Then

$$\frac{\omega d}{2v} = \frac{\omega \lambda}{2c \cdot 2} = \frac{2\pi \lambda}{\lambda \cdot 4} = \frac{\pi}{2}.$$

Hence,

$$T = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}.$$

Therefore,

$$T_{\text{pillbox}} = \frac{2}{\pi}$$

for a relativistic particle crossing a half-wavelength pillbox gap.

This shows that even an ideal pillbox cavity does not provide the full peak voltage.

More General Definition In a realistic cavity, the longitudinal field is not uniform in z . Then the transit-time factor is defined more generally by

$$T = \frac{|\int E_z(z) e^{i\omega z/v} dz|}{\int |E_z(z)| dz}.$$

This expression accounts for both the spatial field profile and the time variation seen by the particle.

The simple sinc-function result

$$T = \frac{\sin(\omega d/2v)}{\omega d/2v}$$

is the special case for a uniform field over a gap of length d .

2.3.3 RF Cavity as a Damped Oscillator

An RF cavity behaves like a damped harmonic oscillator driven by an external RF source. The cavity field amplitude obeys an equation analogous to

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2x = De^{i\omega t},$$

where α is the damping decrement, ω_0 the natural frequency, and D the drive amplitude. The steady-state response is

$$x = Ae^{i\omega t}, \quad A = \frac{D}{\omega_0^2 - \omega^2 + i2\alpha\omega}.$$

The resonance curve shows the usual dependence of amplitude on frequency, with resonance near

$$\omega_r = \sqrt{\omega_0^2 - 2\alpha^2}.$$

The finite resonance width is determined by damping, and the oscillator phase lags behind the drive by an amount that depends on the detuning. In RF terminology, this phase shift is the **tuning angle**.

The quality factor is

$$Q = \frac{\omega_r}{2\alpha},$$

and the amplitude decay time is

$$t_d = \frac{1}{\alpha} = \frac{2Q}{\omega_r}.$$

For an accelerating cavity, Q is higher when no beam is present (unloaded Q_0) and lower when the beam extracts energy (loaded Q).

2.3.4 Cavity Losses, Shunt Impedance, and Quality Factor

In a normal-conducting cavity, the RF magnetic field induces suRFace currents in the cavity walls. Because the wall conductivity is finite, these currents dissipate power as heat. The fields penetrate only through the **skin depth**

$$\delta_s = \sqrt{\frac{2}{\mu_w \sigma_w \omega}},$$

where σ_w is the wall conductivity and μ_w the wall permeability.

The power dissipated in the cavity walls can be expressed in terms of the effective cavity voltage as

$$P_{\text{cy}} = \frac{V_{\text{cy}}^2}{2R_s},$$

where R_s is the **shunt impedance** of the cavity. A larger shunt impedance corresponds to less wall power for a given accelerating voltage.

The **specific shunt impedance** is

$$r_s = \frac{R_s}{d},$$

and is useful for comparing cavities of different lengths. Approximately, the specific shunt impedance scales like $\sqrt{f_{\text{RF}}}$, favoring higher frequencies, although practical limitations such as aperture, cooling, and beam passage constrain how high the frequency can be chosen.

The energy stored in the TM_{010} cavity mode is

$$W = \frac{1}{2}\epsilon_0 E_{010}^2 da^2 J_1^2(2.405),$$

and the cavity quality factor is defined by

$$Q = \omega \frac{W}{P_{\text{cy}}}.$$

For the pillbox cavity, one finds

$$Q = \frac{d}{\delta_s} \frac{\mu_w}{\mu} \frac{a}{a+d}.$$

The cavity filling time, or field build-up/decay time, is

$$\tau_{\text{cy}} = \frac{2Q}{\omega}.$$

Thus the same Q that describes loss also governs how quickly the cavity responds to changes in RF drive.

2.4 RF Parameters for Circular Accelerators

2.4.1 General Considerations

A circular accelerator requires a number of RF parameters to be chosen: RF frequency, harmonic number, cavity voltage, synchronous phase, and energy acceptance. These choices depend not only on beam dynamics but also on technical and economic constraints. The design must consider beam energy range, beam current, ring circumference, momentum compaction, bending radius, and the required rate of acceleration or compensation for synchrotron-radiation loss.

2.4.2 RF Frequency and Harmonic Number

The RF frequency must satisfy the synchronicity condition:

$$f_{\text{RF}} = hf_{\text{rev}},$$

where h is the **harmonic number**, an integer multiple of the revolution frequency.

For highly relativistic beams, the choice of RF frequency is not fundamentally fixed by beam dynamics and is often driven by a compromise among cavity size, bunch length, beam-environment interactions, space-charge effects, and available RF technology. Lower frequencies give longer bunches, reduced wakefield excitation, and reduced particle density, which can help with intra-beam scattering and space-charge effects, especially in proton and ion machines. However, lower frequencies also imply larger, less efficient cavities and may limit available power sources.

Thus:

- **lower RF frequency** favors longer bunches and weaker collective effects,
- **higher RF frequency** favors shorter bunches and more compact cavities,
- the actual choice is usually a compromise among beam physics, engineering, and cost.

2.4.3 Synchronous Phase and RF Voltage

In a linear accelerator, particles are typically accelerated near the crest of the RF wave. In a circular accelerator, however, **phase focusing** requires operation at a **synchronous phase** ϕ_s away from the crest, so that the effective accelerating voltage is

$$V_a = \hat{V}_{\text{cy}} \sin \phi_s.$$

The peak cavity voltage and synchronous phase are determined by the required energy gain per turn and the desired longitudinal acceptance.

The RF voltage in a circular machine must be large enough to:

1. accept the incoming beam's energy spread at injection,
2. accommodate timing or phase errors of the injected beam,
3. compensate beam loading for high-current beams,
4. contain the equilibrium energy spread, especially in electron machines where synchrotron-radiation quantum excitation produces a Gaussian energy distribution.

In practice, this means the machine must be designed for an **energy acceptance of a few percent**, with the most stringent requirement depending on the accelerator type and application.

2.5 Summary

1. Free plane waves are not suitable for acceleration because they lack longitudinal electric field.
2. Conducting boundaries create waveguide and cavity modes with nonzero E_z .
3. Only **TM modes** are useful for particle acceleration.
4. In smooth waveguides, the phase velocity exceeds c , so direct acceleration requires either structural loading or standing-wave cavities.
5. The **TM₀₁₀** cylindrical cavity mode is the fundamental accelerating mode of a pillbox cavity.

6. The actual energy gain is reduced by the **transit-time factor**, which accounts for the finite cavity crossing time.
7. RF cavities behave like damped driven oscillators characterized by resonance frequency, tuning angle, and quality factor.
8. Power losses in normal-conducting cavities are described by the **shunt impedance** and **quality factor**.
9. In circular accelerators, the RF frequency and voltage are chosen as a compromise among synchronization, energy acceptance, bunch length, collective effects, cavity size, and RF-system practicality.

References

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